The Open University

MS221
Exploring Mathematics

Computer Books A–D

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About MS221

This module, MS221 *Exploring Mathematics*, and the modules MU123 *Discovering mathematics* and MST121 *Using Mathematics* provide a flexible means of entry to university-level mathematics. See the address below for further details.

MS221 uses Mathcad (Parametric Technology Corporation) to investigate mathematical and statistical concepts and as a tool in problem solving. This software is provided as part of MS221.

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1.1

Contents

Computer Book A

Mathematical Exploration

Guidance notes

This computer book contains those sections of the chapters in Block A which require you to use Mathcad. Each of these chapters contains instructions as to when you should first refer to particular material in this computer book, so you are advised not to work on the activities here until you have reached the appropriate points in the chapters.

In order to use this computer book, you will need the following Mathcad files.

Chapter A1
221A1-01 Fibonacci numbers
221A1-02 Linear second-order recurrence sequences
221A1-03 A Fibonacci sunflower (Optional)

Chapter A2
221A2-01 Ellipses in parametric form
221A2-02 Hyperbolas in parametric form
221A2-03 Focus directrix and eccentricity (Optional)

Chapter A3
221A3-01 Isometries and triangles
221A3-02 Isometries and conics
221A3-03 Surface and contour plots (Optional)

Instructions for installing these files onto your computer's hard disk, and for opening them, are given in Chapter A0 of MST121.

The computer activities for Chapter A2 also require you to work with one Mathcad worksheet which you have created yourself.

Activities based on software vary both in nature and in length. Sometimes the instructions for an activity appear only in the computer book; in other cases, instructions are given in the computer book and on screen. Feedback on an activity is sometimes provided on screen and sometimes given in the computer book.

For advice on how each computer session fits into suggested study patterns, refer to the Study guides in the chapters.

Chapter A1, Section 4
Exploring linear second-order recurrence sequences with the computer

Mathcad can quickly produce tables of terms of linear second-order recurrence sequences, and of expressions related to such sequences. These can help you to spot patterns in the sequences, and so form conjectures about them. In this section you will be introduced to the Mathcad techniques needed to produce such tables. You will then use examples of these tables to look for patterns and to form conjectures.

The section ends with an *optional* subsection in which you are invited to explore the relationship between the pattern of florets on a sunflower head and the Fibonacci numbers, using a Mathcad file that plots a sunflower pattern.

Further information on Mathcad features can be found in the MS221 reference manual *A Guide to Mathcad*.

4.1 Computer exploration of Fibonacci numbers

In the first activity, you will see how to use Mathcad to calculate terms of the Fibonacci sequence F_n using the recurrence system

See Chapter A1, Section 2.

$$F_0 = 0, F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n = 0, 1, 2, \ldots), \tag{4.1}$$

and how to display these terms in a convenient table.

Activity 4.1 The Fibonacci recurrence system

The Mathcad worksheets for MS221 have a similar design to those for MST121. Remember to make your own working copies of them.

Open Mathcad file **221A1-01 Fibonacci numbers**. Page 2 of the worksheet describes some key features of MS221 Mathcad worksheets.

When you have finished with page 2, move to page 3 of the worksheet and carry out Task 1.

A solution is given on page 4 of the worksheet.

Comment

◇ Range variables and subscripted variables play essential roles in calculating terms of a recurrence sequence in Mathcad. At the beginning of Task 1, you defined the range variable n as follows:

$$n := 0, 1 \,..\, N - 2.$$

This ensures that the subsequent definitions of the subscripted variables F_0, F_1 and F_{n+2},

$$F_0 := 0, \quad F_1 := 1, \quad F_{n+2} := F_{n+1} + F_n,$$

together define *all* the Fibonacci numbers $F_0, F_1, \ldots, F_N$, *and no others*. Mathcad carries out the final definition here $N - 1$ times, once for each of the values $0, 1, \ldots, N - 2$ in the range of n. As it does so, the subscript $n + 2$ takes each of the values in the range $2, 3, \ldots, N$ in turn.

n	0	1	$\cdots$	$N-2$
$n+2$	2	3	$\cdots$	N

◇ You displayed the Fibonacci numbers $F_0, F_1, \ldots, F_N$ by entering '$F =$'. This causes all the values of the subscripted variable to appear in a table. The table displays the subscripts on the left (if there are more than nine values), and scrolls if there are many values to display.

◇ You defined the variable N to be 20, to calculate the Fibonacci numbers $F_0, F_1, \ldots, F_{20}$. You could calculate more Fibonacci numbers, by increasing the value of N. Mathcad automatically updates the values of all other variables whose definitions appear subsequently and depend on N, for example, the range variable $n := 0, 1 \,..\, N - 2$.

If you increase the value of N, remember to set it to 20 again before continuing.

In Activity 4.1, you saw how to calculate and display terms of a sequence defined by a recurrence system. In the next activity, you will see a simple way to calculate and display terms of a sequence defined by a *formula* involving a variable such as n. This method will be used to produce tables showing the sequences defined by Binet's formula,

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \psi^n) \quad (n = 0, 1, 2, \ldots),$$

and by Binet's approximation, $\phi^n/\sqrt{5}$.

See Chapter A1, Section 3. Recall that

$$\phi = \tfrac{1}{2}(1 + \sqrt{5}),$$
$$\psi = \tfrac{1}{2}(1 - \sqrt{5}).$$

Activity 4.2 Binet's formula and Binet's approximation

Move to page 5 of the Mathcad worksheet, and carry out Task 2. Compare the values calculated using Binet's formula and Binet's approximation with the Fibonacci numbers $F_0, F_1, F_2, \ldots, F_{20}$, which were calculated using the recurrence system (4.1).

You should still be working with Mathcad file 221A1-01.

Comment

◇ The table for Binet's formula shows that the formula does indeed give the same value for F_n as the recurrence system, for $n = 0, 1, \ldots, 20$.

The table for Binet's approximation makes it easy to see that F_n is indeed the nearest integer to the value given by Binet's approximation, for $n = 0, 1, \ldots, 20$. You can also see that, as n increases through the range $0, 1, \ldots, 20$, the values of Binet's approximation alternate above and below the Fibonacci numbers, and become progressively closer to them.

◇ The tables of values on page 5 of the Mathcad worksheet are produced by evaluating formulas involving a range variable which has been defined previously. Each table displays the values taken by the corresponding formula as the range variable takes each value in its range. By default, the table scrolls if there are many values to display, but it can be resized to see all of the values without scrolling.

Here, the range variable is n; its definition, $n := 0, 1 \,..\, N$, appears after the words 'Display range'. The first table is for the formula n and therefore displays $0, 1, \ldots, N$. The second is for F_n, so this table displays $F_0, F_1, \ldots, F_N$, which were all defined on page 3 of the worksheet (and again on page 4). The other two tables are for Binet's formula and Binet's approximation.

Mathcad notes provide extra information about the features and techniques used in the Mathcad files. They are *optional*.

Mathcad notes

◇ Mathcad variables can be defined more than once, so they can take different values at different places in a worksheet. For example, the range variable n is defined as $n := 0, 1 \,..\, N - 2$ on page 4 (to calculate the Fibonacci numbers) and as $n := 0, 1 \,..\, N$ on page 5 (to display them).

This does not apply, however, to range variables.

By default, Mathcad places a green squiggle beneath a variable name that has been redefined. As pointed out on page 4 of the worksheet, this effect may be turned off by pressing `[Ctrl][Shift]r`.

Each definition of a variable is used by Mathcad for all expressions involving the variable which appear below that definition in the worksheet, up to the point where a new definition of the same variable appears or the worksheet ends.

On the same line and to the right counts as 'below', but on the same line and to the left counts as 'above'.

◇ The table of Fibonacci numbers on page 5 of the worksheet appears different in style from that on page 4. The table obtained by entering '$F_n =$' on page 5 is displayed below the expression, whereas that obtained by entering '$F =$' on page 4 is displayed to the right, with an additional grey column to denote the subscript values. These differences (between tables containing the same output values) are due to different Mathcad display defaults being triggered in either case. The table characteristics can be altered by clicking on a value in the table with the *right* mouse button, and then choosing either 'Properties...' or 'Alignment' from the resulting mini-menu.

◇ To enter a Greek letter in Mathcad, you can either click on the appropriate button on the 'Greek' toolbar, or type the equivalent Roman letter and then press `[Ctrl]g`. For example, if you type `a` followed by `[Ctrl]g`, then the 'a' will change into an 'α' (alpha).

Now close Mathcad file 221A1-01.

4.2 Exploring patterns in linear second-order recurrence sequences

In the next series of activities, you are invited to explore the patterns that occur when the terms of linear second-order recurrence sequences are combined in certain ways. First you will be asked to consider two different formulas involving the terms of the Fibonacci sequence, namely

$$\frac{F_{n+1}}{F_n} \quad \text{and} \quad F_n^2 + F_{n+1}^2,$$

and to look for patterns in the sequences defined by these formulas. You will then be asked to explore how these patterns generalise to other linear second-order recurrence sequences.

Mathcad file 221A1-02 has been set up to help you to carry out this investigation – the worksheet displays the sequences defined by the above formulas. However, the notation u_n is used rather than F_n in the worksheet, because you can use the same worksheet to see the sequences corresponding to *any* linear second-order recurrence sequence u_n (subject to the accuracy of Mathcad calculations). You can do this just by

changing the values of the initial terms a and b, and the coefficients p and q, in the recurrence system

$$u_0 = a, u_1 = b, \quad u_{n+2} = pu_{n+1} + qu_n \quad (n = 0, 1, 2, \ldots),$$

which defines the sequence u_n in the file.

Activity 4.3 Patterns in the Fibonacci sequence

Open Mathcad file **221A1-02 Linear second-order recurrence sequences**. The worksheet is set up with $p = 1$, $q = 1$, $a = 0$ and $b = 1$, so u_n is the Fibonacci sequence. It displays the sequences

$$n, \quad u_n, \quad \frac{u_{n+1}}{u_n} \quad \text{and} \quad u_n^2 + u_{n+1}^2.$$

Mathcad displays u_n^2 as $(u_n)^2$.

Look at each of the last two sequences in turn. For each sequence, try to spot a pattern, and hence make a conjecture about a general property of the Fibonacci sequence.

Solutions are given on page 34.

The notation F_n is used in the solutions, since here the sequence *is* the Fibonacci sequence, but you may choose to use u_n instead.

Mathcad notes

- ◇ You may wonder why the range variable n used to display the tables is defined only to take values in the range $1, 2, \ldots, N-1$, when the recurrence system defines all the terms of the sequence from u_0 to u_N. Notice, however, that if we increase the final value of the range variable n from $N-1$ to N, then the last term in the 'ratio' table should be u_{N+1}/u_N. This term cannot be calculated, since the term u_{N+1} was not defined earlier in the file. Also, if the first value of the range variable is 0, then the first term in the 'ratio' table should be u_1/u_0, which cannot be calculated, since $u_0 = 0$.

If you try to display a table which contains a value that cannot be calculated, then Mathcad registers an error and shows no values at all.

- ◇ The default result format has been set so that 'Number of decimal places' is 9. Also, 'Exponential threshold' has been set to 15, so numbers between 10^{-15} and 10^{15} are shown in ordinary decimal notation. (Note that these settings affect only how Mathcad *displays* numbers – it always stores values internally to 15 significant figures for calculation purposes.)
- ◇ All the tables have been resized, to show 19 values without scrolling (for the range $1, 2, \ldots, 20-1$). If N is increased so that there are more values to display, then each table will scroll.

In Activity 4.3, you saw that the sequence F_{n+1}/F_n of ratios of successive terms of the Fibonacci sequence appears to tend to the golden ratio ϕ, and to alternate above and below ϕ. The golden ratio ϕ is one of the roots of the auxiliary equation associated with the Fibonacci sequence.

See Chapter A1, Section 3.

In the next activity you are asked to use the same worksheet to explore the sequence u_{n+1}/u_n of ratios for a number of different linear second-order recurrence sequences u_n. You can change the recurrence sequence in the worksheet by altering the values of p, q, a and b.

You will probably find it helpful to scroll down the worksheet until the line containing the definitions of p, q, a and b is at the top of your screen, before you begin Activity 4.4. This should allow you to see enough values of the sequence to spot any patterns, without having to scroll up and down every time you redefine p, q, a and b.

Activity 4.4 Exploring u_{n+1}/u_n

You should still be working with Mathcad file 221A1-02. Ignore the sequence $u_n^2 + u_{n+1}^2$ for the moment.

For each set of values of p, q, a and b suggested in Table 4.1, look at the sequence u_{n+1}/u_n and make a note of any pattern you observe. Note that the file displays the roots of the auxiliary equation associated with the sequence u_n – which should be helpful! There are some spaces in the table for you to choose your own values of a and b, and you may wish to extend the table on a separate sheet to try further values of p and q.

If an error occurs, then Mathcad highlights the offending expression in red. Clicking on this expression reveals an error message.

Note that the table of ratios may register a 'Divide by zero' error for some combinations of p, q, a and b. This happens here if one of the terms $u_1, u_2, \ldots, u_{N-1}$ of the original sequence is zero.

When you have completed the table, try to make a conjecture about the long-term behaviour of the sequence u_{n+1}/u_n.

Table 4.1

Coefficients		Initial values		Apparent pattern in
p	q	a	b	$u_{n+1}/u_n \quad (n = 1, 2, \ldots)$
1	1	0	1	tends to ϕ, alternates above and below ϕ
1	1	2	1	tends to
1	1			
−1	1	0	1	
−1	1	−1	1	
−1	1			
3	−1			
3	−1			
3	−1			

Solutions are given on page 34.

In Activity 4.3 you met the conjecture that the sequence $F_n^2 + F_{n+1}^2$ consists of every other term of the Fibonacci sequence. In the next activity you are asked to explore the sequence $u_n^2 + u_{n+1}^2$ for a number of different linear second-order recurrence sequences u_n.

Activity 4.5 Exploring $u_n^2 + u_{n+1}^2$

You should still be working with Mathcad file 221A1-02.

For each set of values of p, q, a and b suggested in Table 4.2 on the next page, look at the sequence $u_n^2 + u_{n+1}^2$, and make a note of any pattern you observe.

When you have completed the table, try to make a conjecture about the sequence $u_n^2 + u_{n+1}^2$ which is more general than the above conjecture about $F_n^2 + F_{n+1}^2$, in cases where $p = 1$, $q = 1$ and either $a = 0$ or $b = 0$.

If you have time, you may also like to look at what happens if you vary p.

Table 4.2

Coefficients p	q	Initial values a	b	Apparent pattern in $u_n^2 + u_{n+1}^2 \quad (n = 1, 2, \ldots)$
1	1	0	1	it is every other term of u_n, starting with u_3
1	1	0	2	
1	1	0	3	
1	1	1	0	
1	1	2	0	
1	1	3	0	

Solutions are given on page 34.

Now close Mathcad file 221A1-02.

4.3 Fibonacci numbers in sunflowers (Optional)

The Mathcad file associated with this subsection allows you to investigate the close association between the pattern of florets on a sunflower head and the Fibonacci numbers.

Do not be tempted to spend too long on this subsection!

You may remember that the florets of a sunflower develop from tiny lumps called primordia, which are created one after another on the edge of a disc in the centre of the sunflower head and then move outwards. In most sunflowers, the angle between successive primordia is about 137.5°. This is very close to $360(1 - 1/\phi)°$, where ϕ is the golden ratio.

See Chapter A1, Subsection 2.2.

The Mathcad file simulates the process that takes place on a sunflower head, and the pattern that it produces is shown in Figures 4.1 and 4.2. The file allows you to see what happens if you choose angles other than 137.5°. It also allows you to investigate the spirals that appear in the sunflower pattern. Some clockwise spirals are highlighted in Figure 4.1, and some anticlockwise spirals are highlighted in Figure 4.2. We say here that a spiral is clockwise if your finger moves clockwise when tracing it inwards towards the centre, and anticlockwise otherwise.

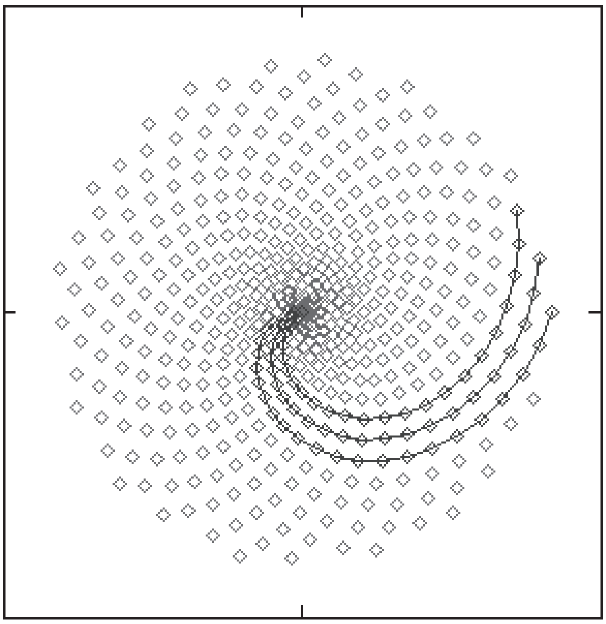


Figure 4.1 Clockwise spirals

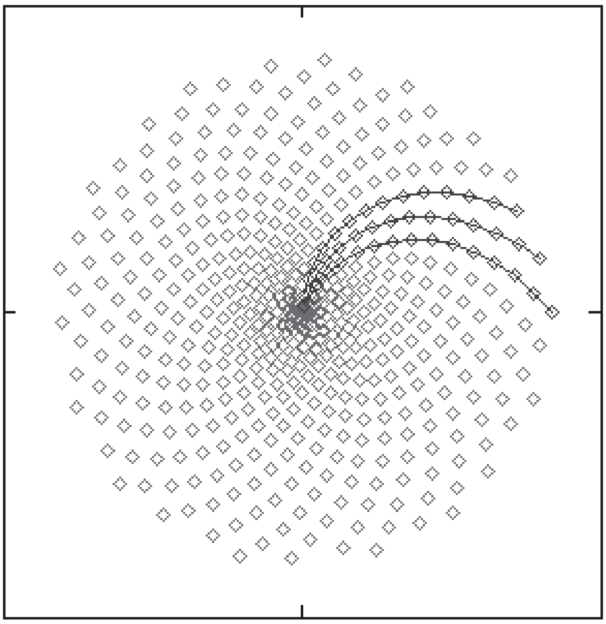


Figure 4.2 Anticlockwise spirals

Activity 4.6 A Fibonacci sunflower (Optional)

Open Mathcad file **221A1-03 A Fibonacci sunflower**, and experiment in the ways described in the worksheet.

Comment

It seems that most choices of the angle between successive points produce a less well-packed pattern than that produced by the angle 137.5°.

In the second part of the investigation, the angle between successive points is left fixed as 137.5°. It appears that highlighting every fth point produces a visible spiral if f is a Fibonacci number (other than 1, 2 or 3), but that no such spiral is produced for most other numbers f.

It seems also that as f increases through the Fibonacci sequence, the spirals produced are alternately clockwise and anticlockwise, and progressively less 'curly'.

Now close Mathcad file 221A1-03.

It is indeed true that Fibonacci numbers always give rise to spirals in a 137.5° sunflower pattern in the way that we have described, and that these spirals have the properties mentioned in the comment above. If you are interested in the reasons behind these facts, you may like to read the explanation given at the end of this subsection.

Figures 4.1 and 4.2 show some of the spirals corresponding to $f = 21$ and $f = 34$.

For each Fibonacci number f with $f \geq 5$, the spiral shown in the Mathcad file is just one of a set of spirals which do not intersect each other but together contain all the points in the sunflower pattern. The number of spirals in the set is f. In any particular sunflower pattern, just some of these sets of spirals are obvious to the eye – the ones in which successive points in each spiral are close together.

This botanical information was kindly provided by Prof. Ralph O. Erickson from the University of Pennsylvania.

The Lucas sequence is $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$. See Chapter A1, Exercise 3.2.

You may be wondering how commonly real sunflowers display the type of pattern that we have explored. Research has established that approximately 95% of sunflowers are 'Fibonacci', where the angle between successive primordia is about 137.5°. Approximately 5% are 'Lucas', where the angle between successive primordia is about 99.5°, and the numbers of spirals are Lucas numbers. A small proportion of sunflowers do not fall into either of these two categories. Research also shows that successive primordia may appear in clockwise or anticlockwise order; 50% of sunflowers are clockwise and 50% anticlockwise.

Explanation of the spirals observations

The key to proving that the observations in the Comment following Activity 4.6 are true in general turns out to be the identity

$$F_{n+1} - \phi F_n = \psi^n, \quad \text{for } n = 0, 1, 2, \ldots, \tag{4.2}$$

which can be derived from Binet's formula, as you will see in Section 5.

All angles in this discussion are measured in degrees.

The sunflower pattern in the Mathcad file is produced by plotting 500 points. The first point is the one on the extreme right-hand side, and each subsequent point is plotted at an angle of approximately $360(1 - 1/\phi)$ clockwise of the previous point, and a small distance nearer the centre.

Since

$$360\left(1-\frac{1}{\phi}\right)=360-\frac{360}{\phi},$$

the *anticlockwise* angle from each point to the next point in the sunflower pattern is $360/\phi \simeq 222.5$. Suppose that we highlight every F_nth point, where F_n is a chosen Fibonacci number. Then the anticlockwise angle from each highlighted point to the next highlighted point is

$$F_n \times \frac{360}{\phi}.$$

By equation (4.2) (with n replaced by $n-1$), this angle is equal to

$$(\phi F_{n-1}+\psi^{n-1})\times\frac{360}{\phi}=(F_{n-1}\times 360)+\left(\frac{\psi^{n-1}}{\phi}\times 360\right).$$

The first term in the expression on the right, $F_{n-1}\times 360$, corresponds to F_{n-1} complete turns, and so the anticlockwise angle from each highlighted point to the next is just

$$\frac{\psi^{n-1}}{\phi}\times 360.$$

Now $\psi = -0.618\ldots$, so ψ^{n-1} alternates in sign and tends to 0 as n tends to infinity. It follows that the above sequence of angles behaves in the same way. Its terms for $n = 2, 3, \ldots, 10$ are shown to three decimal places in the table in the margin.

n	F_n	$\frac{\psi^{n-1}}{\phi}\times 360$
2	1	−137.508
3	2	84.984
4	3	−52.523
5	5	32.461
6	8	−20.062
7	13	12.399
8	21	−7.663
9	34	4.736
10	55	−2.927

Thus, if n is even and large enough, then each highlighted point after the first is the same *small* number of degrees *clockwise* of the preceding highlighted point. Since it is also slightly closer to the centre, a clockwise spiral is produced. This is illustrated in Figure 4.1, which includes the spiral starting from the first point in the pattern when $n = 8$. There is a similar spiral starting from each of the first F_8 points, making a set of $F_8 = 21$ clockwise spirals in all, and three of these are shown in the figure. Similarly, if n is odd and large enough, then each highlighted point after the first is the same *small* number of degrees *anticlockwise* of the preceding highlighted point, and an anticlockwise spiral is produced. Figure 4.2, where $n = 9$, shows three of the set of $F_9 = 34$ anticlockwise spirals.

Remember that the *first* highlighted point is the one furthest from the centre of the sunflower pattern.

So for all large enough Fibonacci numbers F_n there are F_n spirals (provided that there are enough points in the sunflower pattern!), and the spirals are clockwise or anticlockwise according as n is even or odd. Also, since the magnitude of the angle between successive highlighted points decreases as n increases, bigger Fibonacci numbers produce less 'curly' spirals.

Chapter A2, Section 6
Conics on the computer

In this section you will learn how to plot conics on the computer, using parametric representations.

At the end of the section, there is an *optional* subsection in which you are invited to use Mathcad to explore how the shape of a conic alters as its eccentricity changes.

6.1 Ellipses

See Chapter A2, Subsection 5.1.

The standard parametrisation of the ellipse is

$$x = a\cos t, \quad y = b\sin t \quad (0 \le t \le 2\pi).$$

This is a parametrisation of the ellipse in standard position, with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{where } a \ge b > 0).$$

In the first activity you will plot ellipses in standard position, and a translated ellipse.

Activity 6.1 Plotting ellipses from parametric equations

These techniques are used in the computer work for Chapter A2 of MST121, so you may be familiar with them already.

Open Mathcad file **221A2-01 Ellipses in parametric form**. Page 2 of the worksheet describes some basic Mathcad techniques for producing graphs.

When you have finished with page 2, work through pages 3 and 4 of the worksheet, carrying out Tasks 1 and 2.

Comments are provided in the worksheet, and there is a further comment below.

Comment

When you enter expressions involving a range variable in the x- and y-axis placeholders of a graph, Mathcad plots one point for each value of the range. By default, Mathcad joins these points together with line segments, to produce a continuous curve. The result is an approximation to the true curve; the more points are plotted, the more accurate is the approximation.

As a rule of thumb, the range variable used to plot a curve should have a step size which ensures that at least 100 points are plotted. For the ellipses, the graph range $t := 0, 0.01 \,..\, 2\pi$ (a step size of 0.01) was used, which gives over 600 points. Note that the smaller you make the step size, the longer Mathcad will take to plot the graph, and once you reduce the step size beyond a certain level there will be no noticeable improvement in the shape of the curve. It is usually not worth plotting more than 1000 points.

Mathcad notes

The technique used to plot two curves on a graph (such as the two ellipses on page 4 of the worksheet) can be extended to plot three, four, or more curves on the same graph. All the expressions for x should be entered in the x-axis placeholders, separated by commas, and the expressions for y should be entered likewise in the y-axis placeholders. Mathcad plots the first y-axis expression against the first x-axis expression, the second against the second, and so on. When the graph is plotted, the line style and colour used to display each curve appear underneath the expressions on the y-axis.

Remember that Mathcad notes are *optional*.

When a curve is represented by parametric equations, it is sometimes useful to be able to identify the point that corresponds to a particular value of the parameter. In the next activity you will see how to use Mathcad to identify a point on an ellipse.

Activity 6.2 Identifying a particular point on an ellipse

Move to page 5 of the worksheet and carry out Task 3.

You should still be working with Mathcad file 221A2-01.

A solution and comments are provided on page 6 of the worksheet, and there are two further comments below.

Comment

- ◇ If you enter expressions involving an ordinary, single-valued variable (instead of a range variable) in the x- and y-placeholders of a graph, then the result is a plot of a single point. The point can be seen only after the graph trace has been formatted to display it as a symbol. This is the technique that was used to identify a particular point on the ellipse.
- ◇ As the value of T increases through the range $0 \le T \le 2\pi$, the point $(8\cos T, 5\sin T)$ travels once anticlockwise around the ellipse, starting and finishing at $(8, 0)$. In particular, $T = \frac{1}{2}\pi$ corresponds to the point $(0, 5)$, $T = \pi$ to $(-8, 0)$, and $T = \frac{3}{2}\pi$ to $(0, -5)$.

Now close Mathcad file 221A2-01.

6.2 Parabolas

The standard parametrisation of the parabola is

$$x = at^2, \quad y = 2at.$$

See Chapter A2, Subsection 5.1.

This is a parametrisation of the parabola in standard position with equation

$$y^2 = 4ax \quad (\text{where } a > 0).$$

In Activity 6.3 you are asked to use Mathcad to plot such a parabola, with $a = 1$. There is no prepared Mathcad file for this activity – you are asked to create your own Mathcad worksheet. As with every worksheet that you create, it is a good idea to include some text, to make it more comprehensible to a reader such as your tutor (or yourself at a later date!).

Activity 6.3 Plotting a parabola from parametric equations

See *A Guide to Mathcad* if you require more details on creating and editing your own worksheets.

The instructions below tell you how to create a Mathcad worksheet containing a plot of the parabola with parametrisation $x = t^2$, $y = 2t$.

Part (b) describes how to enter a title in the worksheet, and you should also enter any other text that you think is appropriate. For example, when you define a variable, it is helpful to include some text nearby which indicates what the variable represents.

If you have just started Mathcad running, then there is no need to do this, as it automatically starts with a new (Normal) worksheet.

(a) Begin by creating a new worksheet, as follows. Select the **File** menu and choose **New...** . In the list of templates that appears, **Normal** should be selected by default. If not, click on it. Then click on the **OK** button to create a new (Normal) worksheet. (Alternatively, type `[Ctrl]n`, or click on the **New** button on the standard toolbar.)

Before you enter each new item in the worksheet, you should position the red cross cursor in an appropriate place. However, anything that you enter can be moved, or deleted, if you wish.

(b) Enter a title at the top of your worksheet. To do this, first select the **Insert** menu and then choose **Text Region**. (Alternatively, type a double-quote `"`, given by `[Shift]2`.) Then type a suitable title – for example, `Graph of parabola` – in the text box. To finish, click anywhere outside the text box or press `[Ctrl][Shift][Enter]`. If you need to edit the text later, simply click on it.

(c) Define a range variable t going from -5 to 5 in steps of 0.1. You can use the buttons on the 'Calculator' and 'Matrix' toolbars to do this, or just type `t:-5,-4.9;5`. Remember that the second number in the definition is the 'next value' in the range, not the step size.

You can click on the button on the 'Graph' toolbar, or type `@`, or use the **Insert** menu, **Graph ▶ X-Y Plot** .

(d) Create your graph, positioning it below the definition of the range variable. To plot the parabola, enter t^2 in the x-axis placeholder (for example, type `t^2`) and $2t$ in the y-axis placeholder (type `2*t`).

Fix the scales of your graph from 0 to 20 on the x-axis, and from -10 to 10 on the y-axis, and resize the graph appropriately.

(e) Select the **File** menu and use **Save As...** to name and save your worksheet. (You will be working on it again in Activity 6.4.)

Comment

A solution is shown below.

Mathcad text may be made bold, or formatted in other ways, by selecting it and using the 'Formatting' toolbar.

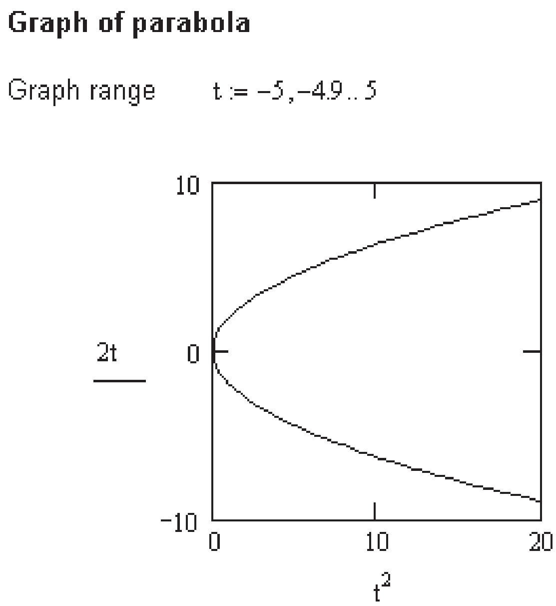


Figure 6.1 The parabola $y^2 = 4x$ in Mathcad

- ◇ After the graph scales have been fixed as described in part (d) of the activity, the graph is 20 units wide and 20 units high. It should therefore be resized to make the graph box square, as shown above.
- ◇ When the axis scales are fixed as described, there are a few points on the parabola that Mathcad calculates but does not plot. For example, for the final value in the graph range, $t = 5$, Mathcad calculates the point $(25, 10)$. This point lies beyond the right-hand edge of the graph, so it is not plotted. When you use Mathcad to plot a graph, it does not matter if some points lie outside the graph box in this way. However, it is preferable (if not always possible!) to choose a combination of graph range and axis scales which avoids the calculation of a large number of points that lie outside the graph box. You should also try to avoid the calculation of points that lie a long way outside the graph box.

In the next activity you are asked to identify a particular point on the parabola that you plotted in Activity 6.3.

Activity 6.4 Identifying a particular point on a parabola

Identify a particular point on the parabola $x = t^2$, $y = 2t$, by plotting a second trace consisting of a single point $(T^2, 2T)$, as follows.

You should still be working with the Mathcad file that you created in Activity 6.3.

(a) First define a variable T with value -4; make sure that this definition is positioned above your graph.

You may wish to make space for this new definition. To do so, position the red cross cursor where you want to insert some blank lines into the worksheet, then press the `[Enter]` key repeatedly until the space created is adequate for your needs. (Each key press will insert one blank line.)

(b) Position the vertical editing line at the right-hand end of the expression t^2 on the x-axis of your graph, with the horizontal editing line under the whole expression. (One way to do this is to click anywhere on the expression, then press `[Space]` and `[Insert]`, as necessary, to select it all.) Type a comma, then enter T^2 (for example, type `,T∧2` – make sure that you type a capital T here).

In the same way, position the vertical editing line at the right-hand end of the expression $2t$ on the y-axis and type a comma, then enter $2T$ (type `,2*T`).

(c) Format the second trace to display the identified point as a box symbol, as follows. Click in the graph to select it, and choose **Graph ▶ X-Y Plot...** from the **Format** menu (or just double-click in the middle of the graph) to bring up the 'Formatting Currently Selected X-Y Plot' option box. Next choose the 'Traces' tab, and click beneath the heading 'Symbol' in the second row (labelled 'trace 2' on the left). Then click on the arrowhead button which appears, and from the drop-down selection choose the box symbol (third one down). Scroll to the right within the option box and, beneath the heading 'Type', alter the entry in the second row from **lines** to **points** (again via a drop-down selection). Click on the **OK** button to finish.

(d) Save your worksheet.

You may now like to increase the value of T and check that the point $(T^2, 2T)$ appears where you expect.

Comment

A solution is shown below.

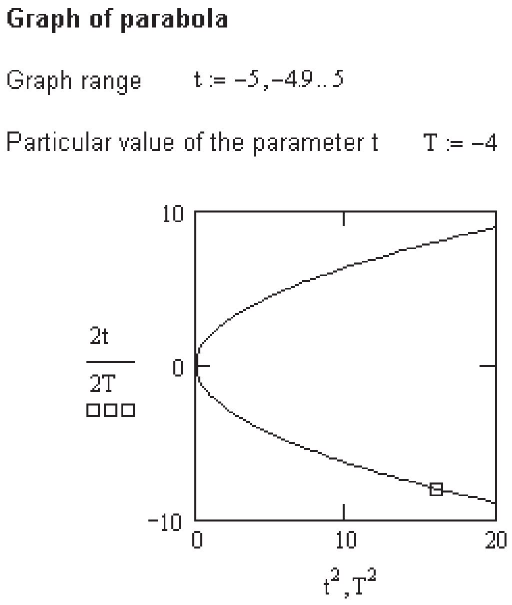


Figure 6.2 Identifying a particular point on $y^2 = 4x$

As T increases through the negative numbers towards 0, the point $(T^2, 2T)$ moves along the lower arm of the parabola towards the origin; when $T = 0$ it is at the origin; and as T increases through the positive numbers it moves away from the origin along the upper arm of the parabola.

If you found that you chose a value for T which caused the identified point to disappear from the graph, this was probably because the point lay outside the graph box! You can display its coordinates by evaluating T^2 and $2T$ in the worksheet. You can see more of the parabola, and see the identified point for more values of T, by changing the graph range and axis scales appropriately.

If you have made unsaved changes then you will see a dialogue box that asks whether you wish to save the changes.

Now close the Mathcad file that you have created.

6.3 Hyperbolas

The standard parametrisation of the hyperbola is

$$x = a\sec t, \quad y = b\tan t \quad (-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi).$$

This is a parametrisation of the hyperbola in standard position, with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (\text{where } a, b > 0).$$

In the next activity you are asked to plot such a hyperbola, with $a = 1$ and $b = 2$. Plotting a hyperbola is a little more tricky than plotting an ellipse or a parabola, because you have to plot the two branches as separate traces.

You will use one range variable, tr, to plot the right-hand branch of the hyperbola, which corresponds to the range $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$, and a second range variable, tl, to plot the left-hand branch of the hyperbola, which corresponds to the range $\frac{1}{2}\pi < t < \frac{3}{2}\pi$.

Activity 6.5 Plotting a hyperbola from parametric equations

Open Mathcad file **221A2-02 Hyperbolas in parametric form**, and carry out Task 1 on page 2 of the worksheet.

Comments are provided in the worksheet, and there are further comments below.

Comment

- ◇ The graph ranges $tr := -1.55, -1.54 \,..\, 1.55$ and $tl := 1.59, 1.60 \,..\, 4.69$ were chosen to ensure that no value of tr or tl is very close to $-\frac{1}{2}\pi$, $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$. This avoids the calculation of points with very large coordinates.
- ◇ You may think that the initial graph shown in the comments in the Mathcad worksheet looks rather strange. This arises from the fact that if you do not enter numbers in the axis limit placeholders when you plot a graph, then by default Mathcad chooses limits which ensure that *all* the points that it calculates are plotted; it also chooses them to be appropriately round numbers. In this case, the endpoints of the portion of the hyperbola corresponding to the ranges of tr and tl are

$$(\sec(-1.55), 2\tan(-1.55)) \simeq (48.1, -96.2),$$
$$(\sec 1.55, 2\tan 1.55) \simeq (48.1, 96.2),$$
$$(\sec 1.59, 2\tan 1.59) \simeq (-52.1, -104.1),$$
$$(\sec 4.69, 2\tan 4.69) \simeq (-44.7, 89.3).$$

  When the axis limits are set closer to zero, as suggested in the worksheet, the part of the curve near the origin can be seen more clearly, and the shape appears much more familiar!

Mathcad notes

The two-letter names tr and tl were used for the range variables because the 'r' and 'l' suggest their use in plotting the right-hand and left-hand branches of the hyperbola. However, we could equally well have used single-letter names, such as t and u, or names consisting of a letter followed by a number, such as $t1$ and $t2$. Note that care is needed when you use the letter 'l' in Mathcad, as it looks very like the number '1'! In general, Mathcad variable names can be any combination of letters and numbers, but they must start with a letter.

In the next activity you are asked to add graph axes and asymptotes to the hyperbola that you plotted in Activity 6.5.

The equations of the axes are $x = 0$ and $y = 0$. These can be plotted using the parametric equations

$$x = 0,\; y = s \quad \text{and} \quad x = s,\; y = 0,$$

respectively.

The equations of the asymptotes of the hyperbola that you plotted in Activity 6.5 are $y = 2x$ and $y = -2x$. These can be plotted using the parametric equations

$$x = s,\; y = 2s \quad \text{and} \quad x = s,\; y = -2s,$$

respectively.

Activity 6.6 Adding graph axes and asymptotes

You should still be working with Mathcad file 221A2-02.

Move to page 3 of the worksheet and carry out Task 2.

Comments are provided in the worksheet.

Mathcad notes

◇ There are four ways to add axes to a Mathcad graph: you can plot extra lines, format the graph to turn 'Show markers' on, format the graph to turn 'Grid lines' on, or format the graph to use the 'Crossed' axis style. You used the first of these in this activity. Details of the other three methods can be found in *A Guide to Mathcad*.

◇ Note that the lines plotted for the axes may not align exactly with the tick marks for 0 at the edges of the graph box. The computer screen is made up of individual pixels (dots) which affect how accurately graphical information can be displayed, and there may be a tiny gap (of one or two pixels) between the lines and tick marks here.

In the next activity you will explore how the shape of the hyperbola changes as you vary a and b.

Activity 6.7 Exploring the shape of the hyperbola

You should still be working with Mathcad file 221A2-02.

Move to page 4 of the Mathcad worksheet. This page contains a plot of the hyperbola with parametrisation

$$x = a\sec t, \quad y = b\tan t \quad (-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi),$$

together with its asymptotes and the graph axes. It is set up so that you can vary the values of a and b.

(a) Keep a fixed at the value 1, and try the values 1, 2 and 3 for b. Describe the effect on the hyperbola.

(b) Keep b fixed at the value 1, and try the values 1, 2 and 3 for a. Describe the effect on the hyperbola.

(c) Explain the effects that you saw in parts (a) and (b).

(d) Try the following pairs of values for a and b: $a = 1$, $b = 2$, and $a = 2$, $b = 4$. What is the relationship between the shapes of the two hyperbolas? You may find it helpful to print them out so that you can compare them.

The point with parameter $t = T$ is identified on the graph; you may like to try varying the value of T as well.

Solutions are given on page 35.

Comment

As T increases through the range $-\frac{1}{2}\pi < T < \frac{1}{2}\pi$, the point $(a\sec T, b\tan T)$ moves upwards along the right-hand branch of the hyperbola, lying on the x-axis when $T = 0$. Similarly, as T increases through the range $\frac{1}{2}\pi < T < \frac{3}{2}\pi$, the point $(a\sec T, b\tan T)$ moves upwards along the left-hand branch of the hyperbola, lying on the x-axis when $T = \pi$. Values of T close to $-\frac{1}{2}\pi$, $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$ correspond to points that do not lie within the graph box.

Mathcad notes

- ◇ You may have noticed that both graph scales are fixed from $-S$ to S, where the variable S is defined at the top of page 4 of the Mathcad worksheet. This has the advantage that the graph scales can be altered just by changing the value of S – there is no need for any changes to the graph itself. Moreover, there is no need to alter the range variable s which is used as the graph range to plot the axes and the asymptotes. This is because s is defined as $s := -S, -S + 0.1 \,..\, S$ (that is, s ranges from $-S$ to S in steps of size 0.1), so the values taken by s will automatically be updated if S changes, and the axes and asymptotes will always be drawn to the edges of the graph box. This is a useful general technique.
- ◇ When a Mathcad graph is to include several curves, it is often best to plot them in the order 'least important first'. This is done for the graph that you used in this activity – the axes are plotted as traces 1 and 2, the asymptotes as traces 3 and 4, the branches of the hyperbola as traces 5 and 6, and the identified point as trace 7. The reason is that Mathcad draws trace 1 first, then trace 2 over the top of that, and so on. So a later trace may obscure some of an earlier trace. In fact, the trace order of the graph that you used in this activity does not have much effect, as the branches of the hyperbola cross the x-axis only once, and never cross the asymptotes!

Now close Mathcad file 221A2-02.

6.4 Focus, directrix and eccentricity (Optional)

The final, optional, activity in this section invites you to explore the shape of a conic with a fixed focus and directrix, and variable eccentricity. In particular, you can see the conic change between the three different types, ellipse, parabola and hyperbola, as the value of the eccentricity changes. The Mathcad worksheet also allows you to explore the focus–directrix property of the conics that it plots.

Activity 6.8 Focus, directrix and eccentricity (Optional)

Open Mathcad file **221A2-03 Focus directrix and eccentricity**. The worksheet shows a plot of the conic with focus $(1, 0)$, directrix $x = -1$, and eccentricity e. The eccentricity e is initially set to 0.8, but you can vary its value.

If the 'redefinition warning' has not been turned off (as can be done by pressing `[Ctrl][Shift]r`), then the e here will have a green squiggle placed beneath it. This is because e ($\simeq 2.718$) is a built-in constant.

A point P on the conic is identified by a blue box symbol; you can change its position by changing the value of the variable T. Two line segments meet at P; the length of the horizontal segment is the distance Pd of P from the directrix d, and the length of the other segment is the distance PF of P from the focus F. The two distances PF and Pd, and the ratio PF/Pd, are evaluated at the bottom right of the worksheet.

(a) Vary the value of the eccentricity e, and observe how the shape of the conic changes. Some possible values for e are suggested in the worksheet, and you may like to try others of your own. In particular, you may like to explore the range of values of e for which it is difficult to distinguish by eye whether the conic is an ellipse, hyperbola or parabola.

You may notice that a small gap appears in some cases between the upper and lower halves of the conic. This is explained, along with other aspects of the worksheet, at the end of this subsection.

You may wish to look at a 'small' conic in more detail, or to see more of a conic that extends outside the graph box. You can decrease or increase the part of the plane corresponding to the graph box by changing the value of the variable S.

(b) Vary the value of T (keeping e fixed) and observe how, although the distances PF and Pd vary as the point P moves along the conic, their ratio PF/Pd remains constant – equal to the eccentricity e.

Note that not every value of T corresponds to a point on the conic. In fact, the x-coordinate of the identified point is equal to T, so by looking at the graph scales you can see roughly which values of T correspond to points on the conic. If you choose a value of T that does not correspond to a point on the conic, then no point is plotted.

For example, with $e = 0.8$, you can see that values of T between about 0 and 9 correspond to points on the conic (the exact range is $\frac{1}{9} \leq T \leq 9$).

Note that the graph is set up to identify points P on the upper half of the conic only.

Comment

When e is not much greater than 0, the conic is an ellipse, nearly circular in shape. It is positioned to the right of the y-axis, with two vertices on the x-axis. As e increases through the range $0 < e < 1$, both of these vertices move along the x-axis: the left one moves further left and approaches the origin, while the right one moves further right, with the result that the ellipse becomes increasingly elongated. When $e = 1$, the vertex that was approaching the origin has reached it, while the other vertex has 'disappeared to infinity'. The conic is now a parabola in standard position. As e increases through the range $e > 1$, the vertex near the origin continues to move left along the x-axis, away from the origin, while the other vertex has 'reappeared' on the negative part of the x-axis, and moves right, approaching the first vertex. The conic is a hyperbola, whose asymptotes become increasingly steep.

If you are interested in how the graph in the worksheet associated with Activity 6.8 is achieved, then you may like to read the following explanation. Otherwise, just close Mathcad file 221A2-03.

Explanation of the Mathcad worksheet

The worksheet states that the conic with focus $(1, 0)$, directrix $x = -1$ and eccentricity e is represented by the equation

$$\alpha x^2 - y^2 + 2\beta x + \alpha = 0, \tag{6.1}$$

This equation can be obtained using the method of the solution to Chapter A2, Activity 3.5.

where $\alpha = e^2 - 1$ and $\beta = e^2 + 1$.

If we then set $x = t$ and solve equation (6.1) for y, we obtain $y = \pm\sqrt{\alpha(t^2 + 1) + 2\beta t}$. Thus we can plot the conic on a Mathcad graph using two separate traces: the first with x-coordinate t and y-coordinate $\sqrt{\alpha(t^2 + 1) + 2\beta t}$, and the second with x-coordinate t and y-coordinate $-\sqrt{\alpha(t^2 + 1) + 2\beta t}$. These are traces 5 and 6 in the Mathcad worksheet. Now t is defined to range from $-S$ to S (where S is defined earlier in the worksheet), but not all of these values of t correspond to points on the conic.

So what happens for the other values of t? In fact, t corresponds to a point on the conic if and only if $\alpha(t^2+1)+2\beta t$ is non-negative. If $\alpha(t^2+1)+2\beta t$ is negative, then according to its definition, the y-coordinate of the point to be plotted is the square root of a negative number. This does not make sense, and so Mathcad does not plot a corresponding point on the graph.

The coordinates of the identified point P on the graph are $x = T$, $y = \sqrt{\alpha(T^2+1)+2\beta T}$, so the identified point always lies on trace 5, the upper trace. If T is set to a value that does not correspond to a point on the conic, then no point is plotted.

We end with explanations of three other aspects of the graph.

- The line segments indicating the distances PF and Pd on the graph are drawn by plotting the three points $(u_0, v_0) = (1, 0)$, $(u_1, v_1) = (x, y)$ and $(u_2, v_2) = (-1, y)$, with a line trace to join them together. This is trace 7 on the graph. The calculations are off the Mathcad page to the right.
- The graph range is defined as $t := -S, -0.998S \,..\, S$; that is, t ranges from $-S$ to S in steps of $0.002S$. This gives a way of plotting 1000 points, whatever the value chosen for S.
- In some cases a point where the conic intersects the x-axis does not correspond with a value of t in the chosen graph range. Then Mathcad cannot plot the corresponding point on the x-axis, which may lead to a slight gap between the upper and lower traces for the conic close to that point.

  For example, with $e = 2.7$, the hyperbola cuts the x-axis at $(-\frac{17}{37}, 0)$, but $-\frac{17}{37} = -0.459\ldots$ is not a value taken by t within the graph range. The closest such values are $t = -0.46$, for which no point is plotted, and $t = -0.44$, for which the corresponding points on the conic are approximately $(-0.44, \pm 0.46)$. Hence a gap of height $2 \times 0.46 = 0.92$ appears on the Mathcad graph.

Note that
$$\frac{S-(-S)}{0.002S} = 1000.$$

This assumes that $S = 10$ in the worksheet, so that t takes the values
$$-10, -9.98, \ldots, 10.$$

Now close Mathcad file 221A2-03.

Chapter A3, Section 5
Isometries on the computer

In this section, you will see how Mathcad can demonstrate the effect of applying isometries in $\mathbb{R}^2$. First isometries are applied to triangles. Then Mathcad is used to plot the graphs of conics with xy-terms, by applying isometries to simpler conics.

There is also an *optional* subsection in which you are invited to use Mathcad to explore surface and contour plots of functions of two variables.

5.1 Isometries and triangles

In this subsection you will use Mathcad to explore the effect of isometries on triangles. The notation used in the Mathcad worksheet for this subsection differs in some respects from that used in the earlier sections of Chapter A3. These changes are needed to allow isometries to be implemented and composed effectively in Mathcad.

A vector is a particular type of **matrix** (a rectangular array of numbers). You will study matrices in detail in Block B.

The main difference is that points in $\mathbb{R}^2$ are represented using **vector notation**: for our current purposes, a **vector** is a column of numbers. For example, to define the point P with coordinates $(5, -2)$ as a vector in Mathcad, we use

$$P := \begin{pmatrix} 5 \\ -2 \end{pmatrix}. \tag{5.1}$$

Remember that in Mathcad the first term of a sequence has subscript 0, by default.

You can think of P as representing a sequence of two numbers $P_0 = 5$ and $P_1 = -2$, written in a vertical table. Thus P_0 is the x-coordinate of P and P_1 is the y-coordinate of P.

In Chapter A3, Section 2, we used $t_{2,-1}$ for this translation, but in Mathcad this notation is not convenient.

We define an isometry in Mathcad as a function whose inputs and outputs are vectors. For example, to define the translation t that moves each point two units to the right and one unit down, we use

$$t(P) := \begin{pmatrix} P_0 + 2 \\ P_1 - 1 \end{pmatrix}. \tag{5.2}$$

Once this definition has been made in a Mathcad worksheet, the function t can be applied to any input vector. For example, if P and t have been defined as in equations (5.1) and (5.2), then evaluating $t(P)$ gives $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$. Also, evaluating $t(P)_0$ gives 7 and evaluating $t(P)_1$ gives -3.

You will learn how to enter and edit vectors, and matrices in general, in Block B. Instructions are also given in *A Guide to Mathcad*.

Vector notation is used throughout the Mathcad worksheet for this subsection, but you will not need to edit any vectors in the worksheet, nor to create any new ones.

In the first activity you will see vector notation used to implement a translation in Mathcad and to demonstrate its effect on a triangle.

Activity 5.1 Points, triangles and the translation function

Open Mathcad file **221A3-01 Isometries and triangles**, and move to page 2 of the worksheet. Read this page carefully – it introduces the notation that will be used throughout the worksheet.

To check that you understand the way in which the translation function has been implemented, try changing the values of c and d so that the image triangle is in a different position, for example, with its left-most vertex at the origin.

Comment

◇ The purpose of the 'point function'

$$P(n) := \begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

defined in the worksheet, is to give an easy way to deal with the vertices (x_0, y_0), (x_1, y_1), (x_2, y_2) and $(x_3, y_3) = (x_0, y_0)$ of the triangle. It means that $P(n)$ is the nth vertex, and $t(P(n))$ is the image of the nth vertex under the translation t. The x-coordinate of the nth vertex is then $P(n)_0$, and its y-coordinate is $P(n)_1$. Similarly, the x-coordinate of the image of the nth vertex under t is $t(P(n))_0$, and its y-coordinate is $t(P(n))_1$. You can see these expressions used on the axes of the graph to plot the two triangles.

◇ In the Mathcad graph, the original triangle is plotted as a solid black trace and the image triangle as solid red. If you have difficulty distinguishing between the triangles, or wish to print the graph on a non-colour printer, then you may prefer to change the line style of the image triangle trace from solid to dashed or dotted.

Similar advice applies to many of the Mathcad graphs that you will see in this section. See the solution to Activity 5.5, for example.

Mathcad notes

The graph at the bottom of the Mathcad page has been resized and both scales have been fixed from 0 to 10. It has also been formatted to show grid lines – this was done by switching on 'Grid lines', then switching off 'Auto grid' and setting the 'Number of grids' to 10 for each axis.

The Mathcad notation that you saw in Activity 5.1 is used throughout the rest of the Mathcad worksheet to implement isometries and demonstrate their effects on triangles.

Remember that when plotting the effects of isometries it is important to have equal scales on both axes.

In the next activity you will see the effect of a rotation demonstrated in Mathcad. The rotation is implemented by defining an appropriate value for the angle θ and setting

$$r(P) := \begin{pmatrix} P_0 \cos(\theta) - P_1 \sin(\theta) \\ P_0 \sin(\theta) + P_1 \cos(\theta) \end{pmatrix}.$$

Remember that P_0 and P_1 are the x- and y-coordinates of P.

Activity 5.2 Rotations

You should still be working with Mathcad file 221A3-01.

Move to page 3 of the Mathcad worksheet. A triangle is defined near the top of the page. Further down, the rotation $r = r_\theta$ is defined, and a graph shows the effect of r_θ on the triangle when $\theta = \frac{1}{2}\pi$. You can change the value of θ, and you can enter this angle in either radians or degrees, as explained in the worksheet.

(a) Figure 5.1 shows the original triangle defined in the worksheet, and its images under three rotations, namely $r_{3\pi/4}$, $r_{-\pi/4}$ and $r_{-5\pi/6}$. Try to decide which image corresponds to which rotation. Then confirm your answer by trying each of the three values of θ in turn in the worksheet and checking that the image appears where you expect.

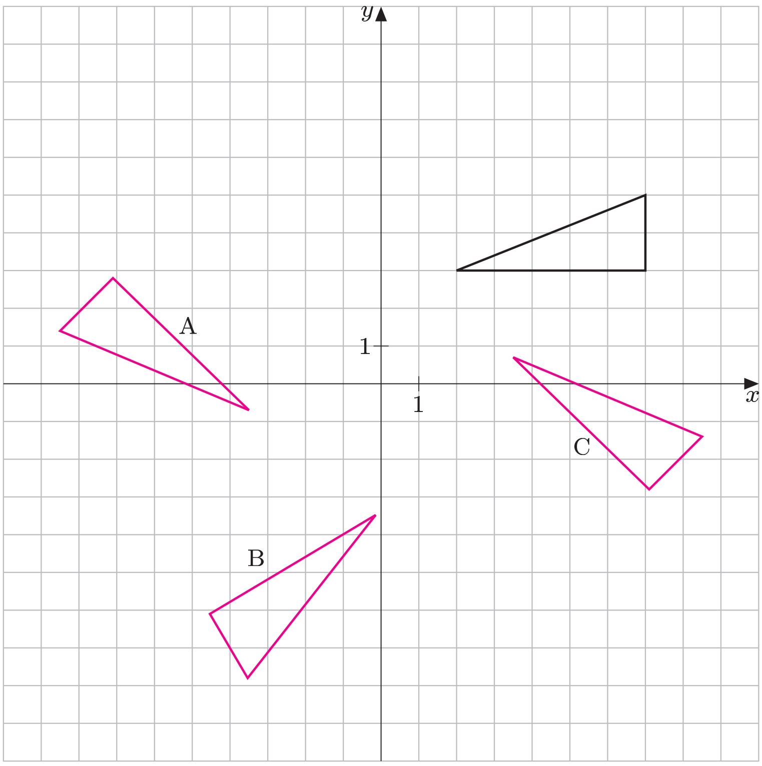


Figure 5.1

(b) Repeat part (a) for Figure 5.2, which shows the images of the triangle under rotations through 60°, −160° and −100°.

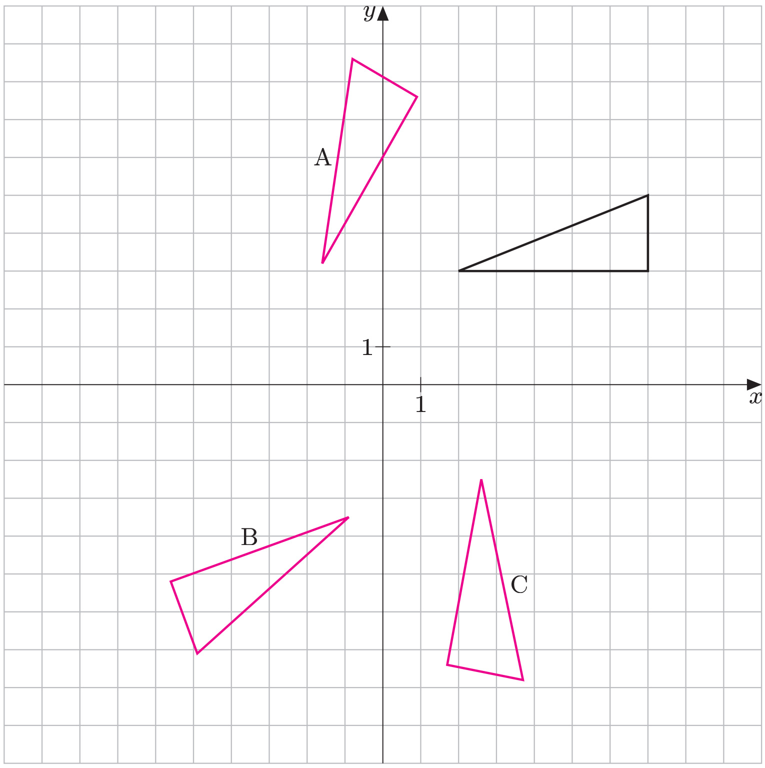


Figure 5.2

Solutions are given on page 35.

Comment

Note that traces 1 and 2 on the graph plot the x- and y-axes, using the parametrisations $x = s$, $y = 0$ and $x = 0$, $y = s$.

Mathcad notes

◇ Note that deg (in lower-case letters) is a built-in Mathcad constant. Its value is 0.017 45 . . . , the number of radians in one degree. For example, evaluating 180 deg gives 3.141 59 . . . . (The constant deg is included in the system of units built into Mathcad, alongside standard SI units, such as m for metre and kg for kilogram. So deg is *not* defined in worksheets where these units are switched off – **Tools** menu, **Worksheet Options...** , 'Unit System' tab, Default Units 'None'.)

◇ You may find it helpful to reformat the graph to add grid lines. To do this, switch 'Grid lines' on, switch 'Auto grid' off, and set 'Number of grids' to 20 for both axes. (You can also alter the colour of the grid lines, by clicking on the coloured rectangle to the right of 'Grid lines' for either axis; the default colour for the grid lines is green.) However, displaying this number of grid lines makes the axis labelling rather overcrowded, and may make the graph less clear if printed.

In the next activity you will see the effect of a reflection demonstrated in Mathcad.

Activity 5.3 Reflections

You should still be working with Mathcad file 221A3-01.

We use θ for the angle of a rotation and ϕ for the angle of a reflection, to avoid confusion.

Move to page 4 of the Mathcad worksheet. This page is set out in a similar way to page 3. The reflection $q = q_\phi$ is defined, and a graph shows the effect of q_ϕ on the specified triangle, together with the line of reflection. You can change the value of ϕ.

(a) Figure 5.3 shows the original triangle defined in the worksheet, and its images under three reflections, namely $q_{5\pi/6}$, $q_{3\pi/4}$ and $q_{3\pi/8}$. Try to decide which image corresponds to which reflection. Then confirm your answer by trying each of the three values of ϕ in turn in the worksheet and checking that the image appears where you expect.

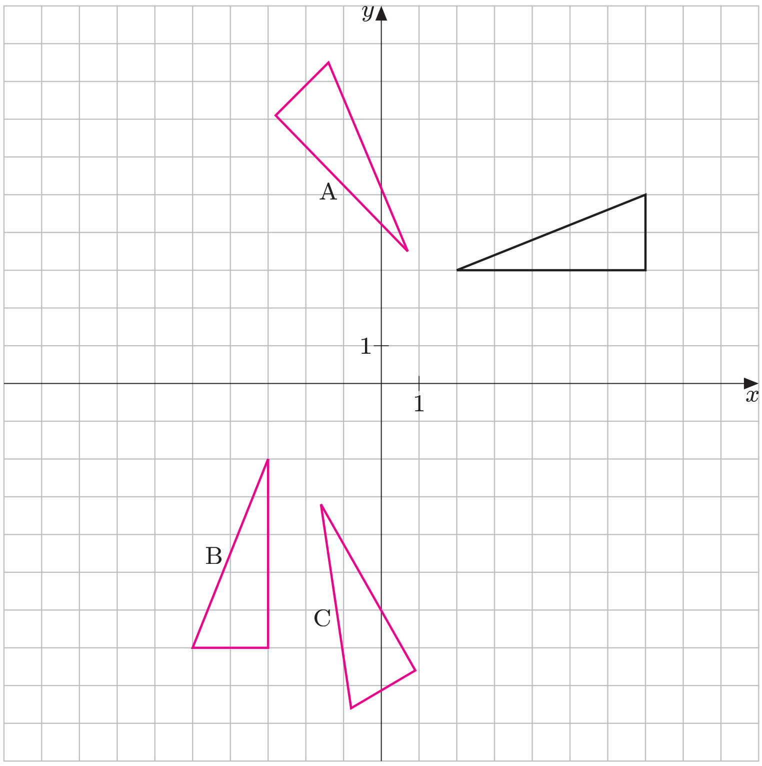


Figure 5.3

(b) Edit the Mathcad page to change the original triangle to the triangle with vertices $(0.9, 6.2)$, $(5.7, 2.6)$ and $(6.3, 8.4)$, and set ϕ to 45°. Then, by experimenting with different values of ϕ, find to the nearest degree the angle ϕ such that the corresponding reflection maps the triangle onto itself.

Recall that you can enter an angle in degrees by using the built-in Mathcad constant 'deg'. For example, for an angle of 180°, enter 180 deg (type `180*deg`).

Solutions are given on page 35.

Comment

The line of reflection is plotted using the parametrisation

$$x = s, \quad y = s\tan\phi.$$

The line making an angle ϕ with the positive x-axis has slope $\tan\phi$. Since only an approximation is used for π, Mathcad gives a (large) finite value for $\tan\left(\frac{1}{2}\pi\right)$.

In the next activity you will see the effect of composites of isometries demonstrated in Mathcad.

Activity 5.4 Composite isometries

Move to page 5 of the Mathcad worksheet. This page demonstrates the effect of the composite $g \circ f$, of an isometry f followed by an isometry g, on a triangle. Initially, f is defined to be the reflection q_ϕ where $\phi = 0$, and g to be the translation $t_{c,d}$ where $c = 1$ and $d = 0$, so $g \circ f$ is a glide-reflection parallel to the x-axis. You can see different glide-reflections by changing the values of ϕ, c and d appropriately (where ϕ is the angle that the line $y = (d/c)x$ makes with the positive x-axis).

You should still be working with Mathcad file 221A3-01.

Figure 5.4 shows the original triangle defined in the worksheet, and its images under three glide-reflections, namely $t_{0,2} \circ q_{\pi/2}$, $t_{-6,-6} \circ q_{\pi/4}$ and $t_{-11,0} \circ q_0$. Try to decide which image corresponds to which glide-reflection. Then confirm your answer by trying each of the three sets of values of ϕ, c and d in turn in the worksheet and checking that the final image appears where you expect.

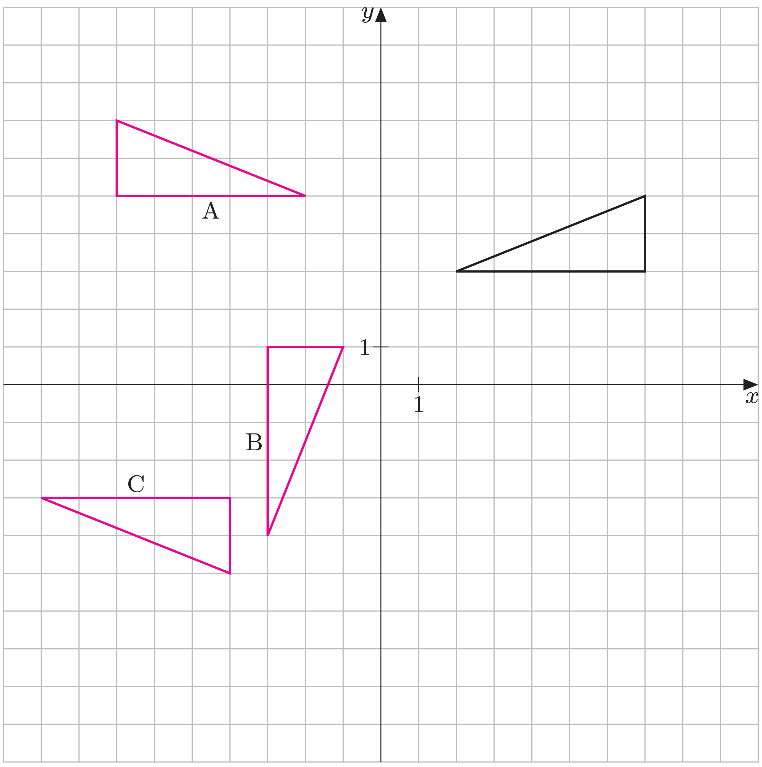


Figure 5.4

Solutions are given on page 35.

Comment

If you wish, you can experiment with the composites of different types of isometries, by changing the definitions of f and g.

Now close Mathcad file 221A3-01.

5.2 Isometries and conics

See Chapter A3, Subsection 4.2.

In the main text, we developed a method of sketching a quadratic curve L with an xy-term. This method involves first determining a conic K with no xy-term that is congruent to L, and a rotation that maps K onto L, then drawing a graph of K, and finally applying the rotation to obtain a graph of L. The final two stages of this process can be carried out using Mathcad, as you will see in this subsection.

The Mathcad worksheet for this subsection uses vector notation in a similar way to the worksheet for Subsection 5.1.

In the first activity you are asked to plot a quadratic curve that is the image under a rotation of an ellipse in reflected standard position. You will use a page of a Mathcad worksheet that has been set up to plot the ellipse with equation

This ellipse is in standard position if $a > b$, and in reflected standard position if $b > a$.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a, b > 0), \tag{5.3}$$

and its image under a rotation r_θ, for values of a, b and θ that are defined in the worksheet and that you can change.

Activity 5.5 A rotated ellipse

Open Mathcad file **221A3-02 Isometries and conics**, read the introduction on page 1 of the worksheet, and move to page 2.

On this page, the ellipse with equation (5.3), and three isometries $t = t_{c,d}$, $r = r_\theta$ and $q = q_\phi$, are defined. An isometry f is initially set to be the rotation r, and the original ellipse and its image under f are plotted on a graph.

You can change the values of a and b to specify different ellipses. You can also change the definition of f to see the image of the ellipse under different types of isometries, and you can change the values of c, d, θ and ϕ to change the isometries themselves.

See Chapter A3, Example 4.1.

The quadratic curve L with equation

$$19x^2 + 6xy + 11y^2 - 40 = 0$$

is the image of the ellipse, in reflected standard position, with equation

$$\tfrac{1}{2}x^2 + \tfrac{1}{4}y^2 = 1$$

Remember that rotations through a positive angle are anticlockwise.

under a rotation of approximately 18° about the origin. Change the values of a, b and θ on page 2 of the worksheet, to obtain the graph of L.

A solution is given on page 35.

Comment

The original ellipse is plotted using the technique that you saw in the computer work for Chapter A2: a large number of points on it are calculated using a suitable range variable u and the usual parametrisation of the ellipse (which remains valid for an ellipse in reflected standard position). However, here we use vector notation to make it easy to deal

In Chapter A2 the name t was used for the range variable, but here we use u.

with the points on the ellipse and their images under the isometry f. Just as the 'point function' P defined the vertices of the triangle in Subsection 5.1, so here we have a 'parametrisation function' p, given by

$$p(u) := \begin{pmatrix} a\cos(u) \\ b\sin(u) \end{pmatrix},$$

which defines points on the ellipse. Each value of u corresponds to a point $p(u)$ on the ellipse, and the image of this point under the isometry f is $f(p(u))$. You can see these expressions used on the graph axes to plot the ellipse and its image under f; as usual, the subscripts 0 and 1 specify x- and y-coordinates, respectively.

Mathcad notes

◇ The square root operator can be obtained by clicking on the appropriate button on the 'Calculator' toolbar, or by typing `\` (a backslash). For example, to obtain $\sqrt{2}$ you can type `\2`.

◇ You cannot use the same name for a variable and a function in a Mathcad worksheet. Here we use t for a translation, so a different name is required for the range variable – we use u.

In the next activity you will use Mathcad to produce a graph of a hyperbola whose equation has an xy-term.

Activity 5.6 A rotated hyperbola

Move to page 3 of the Mathcad worksheet. It is set up in the same way as page 2, but the curve plotted is a hyperbola.

You should still be working with Mathcad file 221A3-02.

The quadratic curve L with equation

$$24x^2 + 20xy - 24y^2 - 13 = 0$$

is the image of the hyperbola with equation

$$2x^2 - 2y^2 = 1$$

You could work this out for yourself by following the strategy given in Chapter A3, Subsection 4.2.

under a rotation of approximately 11° about the origin. Change the values of a, b and θ on page 3 of the worksheet, to obtain the graph of L.

A solution is given on page 35.

We now discuss quadratic curves that are the image under a rotation of a hyperbola in *reflected* standard position.

A hyperbola in reflected standard position does not have parametric equations of the same form as those for a hyperbola in standard position, so the original hyperbola on the Mathcad page will never be in reflected standard position.

Instead, we can start with a hyperbola in standard position, reflect it in the line $y = x$ to obtain a hyperbola in reflected standard position, and then rotate it through an appropriate angle to obtain a graph of the required quadratic curve. You are asked to do this in the next activity.

Activity 5.7 A reflected and rotated hyperbola

See Chapter A3, Activity 4.3.

The quadratic curve L with equation

$$x^2 + 12xy + 6y^2 - 30 = 0$$

is the image of the hyperbola K in reflected standard position with equation

$$-\frac{x^2}{10} + \frac{y^2}{3} = 1,$$

under a rotation of approximately $-34°$ about the origin. This hyperbola is in turn the image of the hyperbola H in standard position with equation

Reflecting in the line $y = x$ corresponds to exchanging the roles of the x- and y-axes.

$$\frac{x^2}{3} - \frac{y^2}{10} = 1,$$

under reflection in the line $y = x$, that is, under $q_{\pi/4}$.

You should still be working with Mathcad file 221A3-02.

(a) On page 3 of the Mathcad worksheet, set $\theta = 0$ (if it is not already set to 0), and change the values of a and b to obtain a graph of H. The original and image hyperbolas will then be superimposed.

(b) In the definition of the isometry f, change r to q; this means that f is the isometry q_ϕ. Then change the value of ϕ to obtain a graph of the hyperbola K.

Note that in order to obtain the graph of L, you only need to set the values of a, b, ϕ and θ, and edit the definition of f as in part (c).

(c) Now edit the definition of the isometry f again so it reads as follows:

$$f(P) := r(q(P)).$$

This means that f is the isometry $r_\theta \circ q_\phi$. Change the value of θ to obtain a graph of L.

Solutions are given on page 36.

Comment

Since reflecting a hyperbola in standard position in the line $y = x$ produces the same image as rotating it about the origin through the angle 90°, an alternative way to obtain a graph of the above quadratic curve L is to begin with the same hyperbola H in standard position, and simply rotate it through $90° + (-34)° = 56°$.

In the next activity you will use Mathcad to produce a graph of a parabola with an xy-term. Here, we need to use a translation as well as a rotation.

Activity 5.8 A translated and rotated parabola

You should still be working with Mathcad file 221A3-02.

Move to page 4 of the Mathcad worksheet. It is set up in the same way as pages 2 and 3, but the curve plotted is a parabola and the isometry f is set initially as a translation.

You could work this out for yourself by following the method explained in Chapter A3, Subsection 4.2. Note that in this case $D \neq 0$ and $E \neq 0$.

The quadratic curve L with equation

$$9x^2 - 24xy + 16y^2 - 10x - 70y - 75 = 0$$

is the image of the quadratic curve K with equation

$$y^2 - 2x - 2y - 3 = 0,$$

which has no xy-term, under a rotation of approximately 37° about the origin. Completing the square in the equation for K, and simplifying, gives

$$(y - 1)^2 - 2(x + 2) = 0,$$

so the curve K is itself the image of the parabola H in standard position with equation

$$y^2 = 2x,$$

under the translation $t_{-2,1}$.

(a) On page 4 of the worksheet, set $c = 0$ and $d = 0$ (if they are not already set to 0) and change the value of a to obtain a graph of H.

(b) Change the values of c and d to obtain a graph of K.

(c) Now edit the definition of the isometry f so that it reads as follows:

$$f(P) := r(t(P)).$$

This means that f is the isometry $r_\theta \circ t_{c,d}$. Change the value of θ to obtain the graph of L.

Solutions are given on page 36.

Now close Mathcad file 221A3-02.

5.3 Surface and contour plots (Optional)

In this subsection you are invited to use Mathcad to explore surface and contour plots of functions of two variables. A contour plot can provide a useful check on the shape and position of a quadratic curve.

See Chapter A3, Subsection 1.3.

Activity 5.9 Surface and contour plots (Optional)

Open Mathcad file **221A3-03 Surface and contour plots**, and read through pages 1 and 2 of the worksheet.

(a) In the definition of the function f on page 2 of the worksheet, change 'd' to 'g' so that the function g is plotted instead of the distance function d.

The quadratic curve with equation

$$4x^2 + 9y^2 - 16x - 18y - 11 = 0$$

is then displayed as the contour with height 0.

This is the ellipse E discussed in Chapter A3, Section 1.

(b) Now use page 2 of the worksheet to verify the shapes of the quadratic curves that you considered in Activities 5.5 to 5.8, as follows.

For each quadratic curve, change the values of the coefficients in the worksheet to the values of A, B, C, D, E and F corresponding to the equation of the curve. Check that the shape and position of the contour with height 0 are the same as those of the curve.

(c) If you would like to see how to create and to format surface and contour plots, then read page 3 of the worksheet.

Now close Mathcad file 221A3-03.

Solutions to Activities

Chapter A1

Solution 4.3

The value of the expression F_{n+1}/F_n appears to tend to the golden ratio ϕ as n increases, and appears to lie alternately above and below ϕ.

(You met this property in Section 2 and will see a proof of it in Section 5.)

The value of the expression $F_n^2 + F_{n+1}^2$ always seems to be a Fibonacci number. These numbers appear in alternate positions in the F_n table. To be precise,

$$\begin{aligned} F_1^2 + F_2^2 &= F_3, \\ F_2^2 + F_3^2 &= F_5, \\ F_3^2 + F_4^2 &= F_7, \\ &\vdots \\ F_9^2 + F_{10}^2 &= F_{19}. \end{aligned}$$

This suggests the conjecture that, for $n = 1, 2, 3, \ldots$,

$$F_n^2 + F_{n+1}^2 = F_{2n+1}.$$

(This conjecture is true. It can be proved using Binet's formula, but we shall not do so.)

Solution 4.4

It appears that the sequence u_{n+1}/u_n always tends to one of the roots of the auxiliary equation, provided that these roots are real. In fact, if the roots are real and distinct, then the sequence seems to tend to the root of larger magnitude, whether this is positive or negative. (The *magnitude* of a real number x is x if $x \geq 0$ and $-x$ if $x < 0$. For example, the magnitudes of 3 and -3 are both 3.) It is natural to conjecture that this happens for any linear second-order recurrence sequence whose auxiliary equation has real roots. Changing the initial values a and b does not seem to affect this long-term behaviour.

(This conjecture is indeed true for all such linear second-order recurrence sequences u_n, but we prove it only for the particular case of the Fibonacci sequence, in Section 5.)

You may also have noticed that although for the Fibonacci sequence the terms of the sequence u_{n+1}/u_n appear to alternate above and below the value to which they tend, this does not happen for every linear second-order recurrence sequence.

(This behaviour is related to whether the roots of the auxiliary equation are of the same sign or of opposite signs.)

Solution 4.5

It appears that if $p = 1$, $q = 1$ and $a = 0$, then the sequence $u_n^2 + u_{n+1}^2$ $(n = 1, 2, 3, \ldots)$ consists of b times every other term of the original sequence u_n, starting with u_3. This suggests the conjecture that if $p = 1$, $q = 1$ and $a = 0$, then, for all b,

$$u_n^2 + u_{n+1}^2 = bu_{2n+1}, \quad \text{for } n = 1, 2, 3, \ldots. \qquad \text{(S3.1)}$$

Similarly, it appears that if $p = 1$, $q = 1$ and $b = 0$, then the sequence $u_n^2 + u_{n+1}^2$ $(n = 1, 2, 3, \ldots)$ consists of a times every other term of the original sequence, starting with u_2. This suggests the conjecture that if $p = 1$, $q = 1$ and $b = 0$, then, for all a,

$$u_n^2 + u_{n+1}^2 = au_{2n}, \quad \text{for } n = 1, 2, 3, \ldots. \qquad \text{(S3.2)}$$

If you tried varying p, then you would have found that this seems to have no effect on the above conjectures; that is, the first conjecture seems to be true if $q = 1$, $a = 0$ and for all p and b, and the second seems to be true if $q = 1$, $b = 0$ and for all p and a.

(The identities in equations (S3.1) and (S3.2) can in fact be deduced from the identity

$$F_n^2 + F_{n+1}^2 = F_{2n+1}, \quad \text{for } n = 1, 2, 3, \ldots,$$

which you met in Activity 4.3. For equation (S3.1), we use the fact that if $p = 1$, $q = 1$ and $a = 0$, then

$$u_n = bF_n, \quad \text{for } n = 1, 2, 3, \ldots.$$

Therefore, for $n = 1, 2, 3, \ldots$,

$$\begin{aligned} u_n^2 + u_{n+1}^2 &= (bF_n)^2 + (bF_{n+1})^2 \\ &= b^2(F_n^2 + F_{n+1}^2) \\ &= b^2 F_{2n+1} \\ &= bu_{2n+1}. \end{aligned}$$

Equation (S3.2) can be proved in a similar way, using the fact that if $p = 1$, $q = 1$ and $b = 0$, then

$$u_n = aF_{n-1}, \quad \text{for } n = 1, 2, 3, \ldots.$$

In fact, there is a general identity

$$qu_n^2 + u_{n+1}^2 = aqu_{2n} + bu_{2n+1}, \quad \text{for } n = 1, 2, 3, \ldots,$$

which holds for all values of p, q, a and b. This can be proved using the closed form of the sequence, but the details are a little involved.)

Chapter A2

Solution 6.7

(a) If a is fixed, then as b increases, the two points where the hyperbola crosses the x-axis remain fixed, and the asymptotes become steeper.

(b) If b is fixed, then as a increases, the two points where the hyperbola crosses the x-axis move away from the origin, and the asymptotes become less steep.

(c) The points where the hyperbola crosses the x-axis are $(\pm a, 0)$, and the asymptotes are $y = \pm(b/a)x$, where $a, b > 0$ (see Section 2). Thus if a is fixed, then the two crossing points remain fixed; if b now increases, then b/a increases and hence the asymptotes become steeper. On the other hand, if b is fixed and a increases, then the crossing points move away from the origin; also b/a decreases and hence the asymptotes become less steep.

(d) The two hyperbolas have the same asymptotes, since they have the same value of b/a. In fact, their shapes are even more closely related: they are similar. The first hyperbola has parametrisation

$$x = \sec t, \quad y = 2\tan t \qquad (-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi)$$

and the second has parametrisation

$$x = 2\sec t, \quad y = 4\tan t \qquad (-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi),$$

so if the point (u, v) lies on the first hyperbola, then the point $(2u, 2v)$ lies on the second.

(In general, any two hyperbolas with the same value of b/a are similar: each is obtained from the other by a scaling with the same scale factor with respect to both axes. This is also true for ellipses.)

Chapter A3

Solution 5.2

(a) A is the image of the triangle under $r_{3\pi/4}$, B is its image under $r_{-5\pi/6}$, and C is its image under $r_{-\pi/4}$.

(b) A is the image of the triangle under a rotation through 60°, B is its image under a rotation through −160°, and C is its image under a rotation through −100°.

Solution 5.3

(a) A is the image of the triangle under $q_{3\pi/8}$, B is its image under $q_{3\pi/4}$, and C is its image under $q_{5\pi/6}$.

(b) Reflection in a line through the origin that makes an angle of approximately 53° with the positive x-axis maps the triangle onto itself.

Solution 5.4

A is the image of the triangle under $t_{0,2} \circ q_{\pi/2}$, B is its image under $t_{-6,-6} \circ q_{\pi/4}$, and C is its image under $t_{-11,0} \circ q_0$.

Solution 5.5

Setting $a = \sqrt{2}$, $b = 2$ and $\theta = 18°$ gives the graph in Figure S3.1, in which L is the dashed trace. Here the axes limits have been changed to make the plot of the ellipses larger.

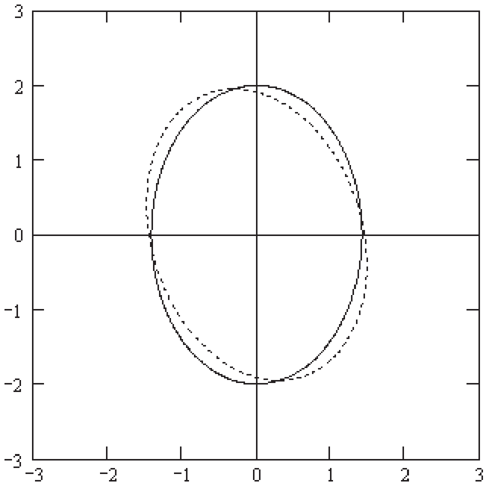


Figure S3.1

Solution 5.6

Setting $a = 1/\sqrt{2}$, $b = 1/\sqrt{2}$ and $\theta = 11°$ gives the graph in Figure S3.2, in which L is the dashed trace.

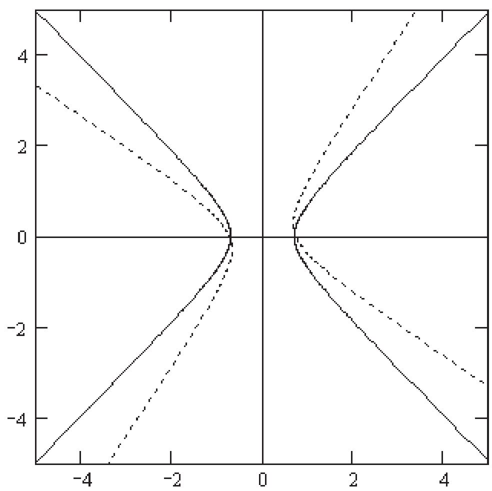


Figure S3.2

Solution 5.7

(a) Setting $\theta = 0$, $a = \sqrt{3}$ and $b = \sqrt{10}$ gives the graph of H, as in Figure S3.3.

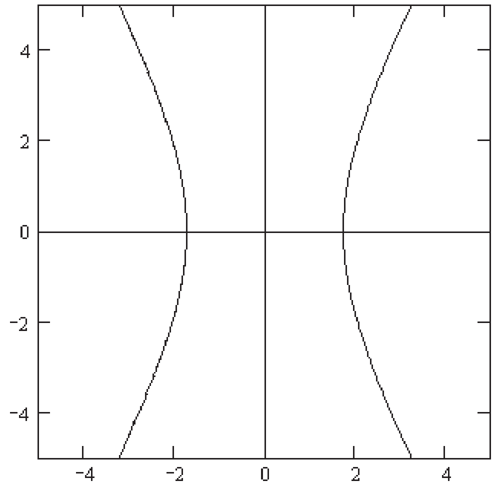


Figure S3.3

(b) Setting $\phi = 45°$ gives the graph in Figure S3.4, in which K is the dashed trace.

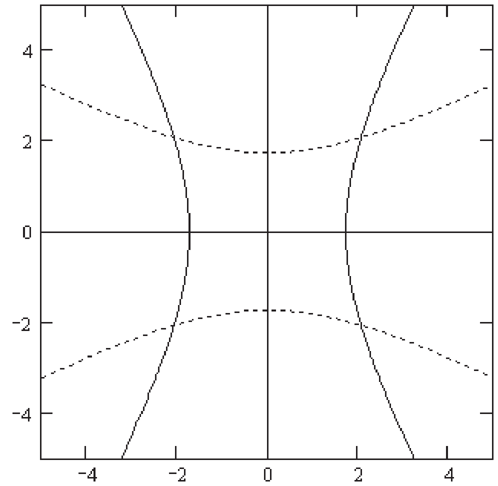


Figure S3.4

(c) Setting $\theta = -34°$ gives the graph in Figure S3.5, in which L is the dashed trace.

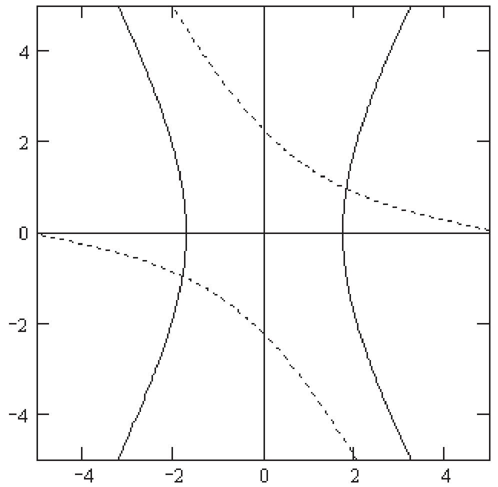


Figure S3.5

Solution 5.8

(a) Setting $c = 0$, $d = 0$ and $a = \frac{1}{2}$ gives the graph of H, as in Figure S3.6.

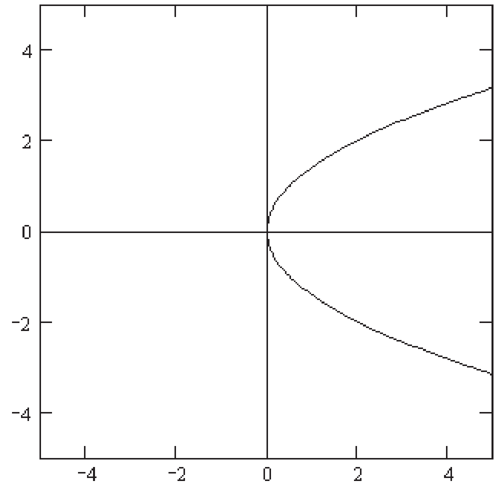


Figure S3.6

(b) Setting $c = -2$ and $d = 1$ gives the graph in Figure S3.7, in which K is the dashed trace.

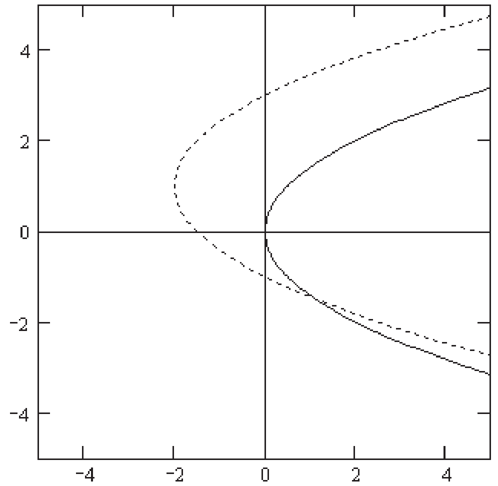


Figure S3.7

(c) Setting $\theta = 37°$ gives the graph in Figure S3.8, in which L is the dashed trace.

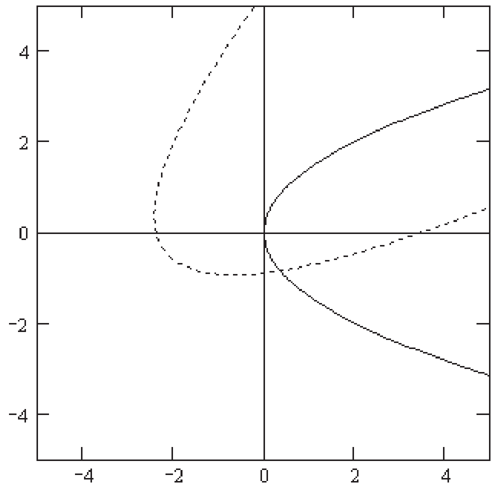


Figure S3.8

Computer Book B

Exploring Iteration

Guidance notes

This computer book contains those sections of the chapters in Block B which require you to use Mathcad. Each of these chapters contains instructions as to when you should first refer to particular material in this computer book, so you are advised not to work on the activities here until you have reached the appropriate points in the chapters.

In order to use this computer book, you will need the following Mathcad files.

Chapter B1

221B1-01 Iterations of f(x)=x^2+c
221B1-02 Overview of iterations of f(x)=x^2+c
221B1-03 Iterating real functions

Chapter B2

221B2-01 Linear transformations
221B2-02 Affine transformations

Chapter B3

221B3-01 Iterating linear transformations
221B3-02 Iterating affine transformations
221B3-03 Iterated function systems (Optional)

Instructions for installing these files onto your computer's hard disk, and for opening them, are given in Chapter A0 of MST121.

Activities based on software vary both in nature and in length. Sometimes the instructions for an activity appear only in the computer book; in other cases, instructions are given in the computer book and on screen. Feedback on an activity is sometimes provided on screen and sometimes given in the computer book.

For advice on how each computer session fits into suggested study patterns, refer to the Study guides in the chapters.

Chapter B1, Section 4
Iterating real functions with the computer

In this section you will explore iteration sequences, using Mathcad worksheets that draw graphical iteration diagrams and calculate related information, such as fixed points and gradients.

In Subsection 4.1, you will explore sequences obtained by iterating quadratic functions of the form $f(x) = x^2 + c$. You will see that many different types of behaviour occur; for example, such sequences can tend to p-cycles for various periods p. This subsection is quite long; if you are short of time, then concentrate on Activities 4.1 and 4.2.

The concept of a p-cycle with $p > 2$ was defined in the main text.

In Subsection 4.2, Mathcad is used to provide an overview of how the behaviour of these iteration sequences changes as c varies, and you will see that the changes in behaviour are surprisingly complicated. Subsection 4.3 is *optional*; it contains a discussion of certain other quadratic iteration sequences. In Subsection 4.4, you will use a Mathcad worksheet that allows you to study the sequences obtained by iterating *any* real function f.

4.1 Sequences obtained by iterating $f(x) = x^2 + c$

In this subsection you will explore sequences obtained by iterating functions of the form $f(x) = x^2 + c$, where c is a parameter whose value can be varied.

You have seen that if we wish to study the long-term behaviour of iteration sequences, then it is useful to find and classify any fixed points and 2-cycles of the function.

The fixed point equation and the 2-cycle equation of the function $f(x) = x^2 + c$ can be solved to give formulas, in terms of c, for the fixed points and for those points that are members of a 2-cycle. The Mathcad worksheet that accompanies this subsection uses these formulas to calculate any fixed points and 2-cycles of f. The number of fixed points and 2-cycles depends on the value of c, but f always has at most two fixed points, and at most one 2-cycle.

Recall that a fixed point a is *attracting* if $|f'(a)| < 1$; *repelling* if $|f'(a)| > 1$; *indifferent* if $|f'(a)| = 1$.

You saw in the main text that a fixed point a can be classified according to the value of $f'(a)$, the gradient of the graph of f at $(a, f(a))$. The gradient formula for the function $f(x) = x^2 + c$ is $f'(x) = 2x$; in particular, for a fixed point a, we have $f'(a) = 2a$.

A 2-cycle a, b is *attracting* if $|f'(a)f'(b)| < 1$; *repelling* if $|f'(a)f'(b)| > 1$; *indifferent* if $|f'(a)f'(b)| = 1$.

Similarly, a 2-cycle a, b of f can be classified according to the value of the gradient product $f'(a)f'(b)$; this is equal to the gradient of the graph of the composite function $f \circ f$ at a, and also at b. For a 2-cycle a, b of the function $f(x) = x^2 + c$, we have $f'(a)f'(b) = (2a)(2b) = 4ab$.

The first activity introduces the Mathcad worksheet that you will be using.

Activity 4.1 Graphical iteration on the computer

Open Mathcad file **221B1-01 Iterations of f(x)=x^2+c**, and move to page 2 of the worksheet. Here N iterations of the function $f(x) = x^2 + c$ are carried out, with initial term x_0. The resulting sequence x_n is displayed on a graphical iteration diagram, and the last eleven terms calculated are listed in a table. Values for the fixed points, the 2-cycle, and the associated gradients and gradient product are displayed (to three decimal places) underneath the table.

Remember to create your own working copy of the Mathcad file.

The page is set up with $c = 0$, so the function is initially $f(x) = x^2$. The fixed point information below the table and diagram shows that this function f has two fixed points, 0 and 1. The values of the gradients show that 0 is attracting whereas 1 is repelling. The 2-cycle variables below indicate that f has no 2-cycle.

The page is also set up with $N = 10$ and $x_0 = 0.8$. In parts (a) to (d) below you are asked to try other values for these two variables.

The variables M and p will be introduced in Activities 4.2 and 4.5, respectively. You should not alter the values of these for now.

(a) Describe the long-term behaviour of the iteration sequence with initial term $x_0 = 0.8$. Then set x_0 to each of the values -0.8, 0 and 1.1 in turn, and in each case describe the long-term behaviour of x_n.

(b) Set $x_0 = 0.999$, which is a number just less than the repelling fixed point. Then set $N = 100$, and describe the long-term behaviour of x_n.

(c) With N still set to 100, set $x_0 = 1.001$, which is a number just greater than the repelling fixed point. Try to explain why Mathcad behaves as it does.

(d) With x_0 still set to 1.001, set $N = 15$, and describe the long-term behaviour of x_n.

Remember that to *set* variables in Mathcad we define them using the symbol := , which can be obtained from the appropriate button on the 'Calculator' toolbar or by typing `:` (a colon, given by `[Shift];`). The symbol = *evaluates* variables in Mathcad.

Comment

(a) When $x_0 = 0.8$, the sequence converges to the fixed point 0, with a staircase pattern of convergence. The same is true when $x_0 = -0.8$.

When $x_0 = 0$, the sequence converges to the fixed point 0. In fact, since 0 is a fixed point, each term of the sequence is equal to 0, which is why no iterations can be seen on the diagram.

When $x_0 = 1.1$, the sequence tends to infinity. Only the first few iterations appear on the diagram, but you can see from the table that the terms of the sequence quickly become very large.

(b) When $x_0 = 0.999$, ten iterations are insufficient to indicate clearly the long-term behaviour of x_n. When the number of iterations is increased to 100, the diagram shows the sequence approaching the fixed point 0. The table provides further evidence for this behaviour, since it shows that the terms x_{90} to x_{100} all have value 0 to three decimal places.

(c) When $x_0 = 1.001$ and $N = 100$, Mathcad marks $f(x_n)$ (above the graph) in red, and neither the table for x_n nor the iteration diagram appears. Clicking on $f(x_n)$ reveals the error message 'Encountered a floating point error.', which indicates that some of the terms of the sequence are too large for Mathcad to cope with. This suggests that the sequence is unbounded. To obtain a clear idea of the behaviour of the sequence, it is helpful to reduce the number of iterations, as suggested in part (d).

(d) The terms in the table quickly become very large, which suggests that x_n tends to infinity; this behaviour is also suggested by the diagram.

Mathcad notes

Remember that Mathcad notes are *optional*.

- ◇ The error that arose in part (c) occurs when the result of a calculation exceeds in magnitude the largest positive number that Mathcad can handle, which is about 10^{307}.
- ◇ The expressions entered on the graph axes to plot the graphical iteration diagram have been 'hidden'. This has been done so that the graph fits on the page! The expressions on the axes of any graph can be hidden by clicking on the graph to select it, then choosing **Graph ▶ X-Y Plot...** from the **Format** menu, selecting the 'Traces' tab and clicking in the check box to 'Hide arguments'.

Double-clicking on the graph gives an alternative approach.

- ◇ The 'staircases' and 'cobwebs' are drawn on the graph by plotting x_{k+1} against x_k using the trace type 'step' (see Figure 4.1).

  Mathcad plots the sequence of points (x_k, x_{k+1}), for $k = 0, 1, 2, \ldots,$ and joins each point (x_k, x_{k+1}) to (x_{k+1}, x_{k+2}) using a horizontal line segment ending at (x_{k+1}, x_{k+1}) followed by a vertical line segment. A separate trace is required to draw the first (vertical) line in the diagram from $(x_0, 0)$ to (x_0, x_1). This is done by plotting $x_1 \cdot (k > 0)$ against x_0 using the default trace type 'lines'. Mathcad assigns the inequality $(k > 0)$ in this expression the value 0 when it is false (for example, when $k = 0$) and 1 when it is true (for example, when $k = 1$).

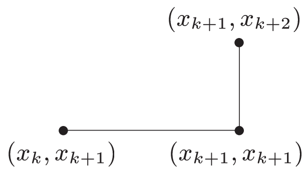


Figure 4.1 Trace type 'step'

In the next activity you will begin to investigate how the sequences obtained by iterating the function $f(x) = x^2 + c$ change as c is varied. So that the investigation is conducted in a systematic manner, you should keep the initial term x_0 set to the same value from now on. A suitable value for x_0 is 0, and this is the value that will be used in the remainder of the investigation. Thus you will be investigating how the behaviour of the iteration sequence

$$x_0 = 0, \quad x_{n+1} = x_n^2 + c \quad (n = 0, 1, 2, \ldots) \tag{4.1}$$

alters as c is varied. Throughout Subsections 4.1 and 4.2, the notation x_n will always mean a sequence of the form defined in equations (4.1).

For some values of c, the long-term behaviour of the sequence x_n may not be immediately obvious. The next activity introduces some ways in which you can adjust the values of variables on the Mathcad page to help to determine the long-term behaviour.

Activity 4.2 Exploring iteration sequences

You should still be working with Mathcad file 221B1-01.

In each of parts (a) to (e) you are asked to determine the long-term behaviour of the sequence x_n for a particular value of c, and then for two further values of c close to the first value. All of these values are marked on the number line in Figure 4.2, on the next page.

You looked at the case $c = 0$ in Activity 4.1(a).

An 'F' has been marked above the number 0 on the number line, to indicate that when $c = 0$ the sequence tends to a fixed point. As you work through the activity, record your findings for the other values of c in a similar manner. Use a '2' if the sequence seems to tend to a 2-cycle, a '3' for a 3-cycle, and so on. If the sequence seems to tend to infinity, then use the symbol '∞'. If it seems to be chaotic, that is, if it does not appear to settle down to any steady long-term behaviour, then use the letter 'X'.

First ensure that $x_0 = 0$. It is convenient to set the number of iterations and graph scale initially as follows: $N = 10$, $s1 = -2$ and $s2 = 2$.

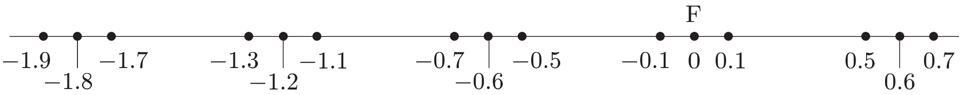


Figure 4.2 Values of c on the number line

(a) Set $c = 0.6$.

Record (on Figure 4.2) the long-term behaviour of the iteration sequence. Then repeat the process with $c = 0.7$ and $c = 0.5$.

(b) Set $c = 0.1$.

You may find it helpful to rescale the diagram, so that you can see the behaviour near the fixed point in more detail; try setting $s1 = -0.2$ and $s2 = 0.2$.

Record the long-term behaviour of the sequence. Then repeat the process with $c = -0.1$.

(c) Reset $s1 = -2$ and $s2 = 2$. Set $c = -0.6$.

You may find it helpful to increase the number of iterations and rescale the diagram; try $N = 100$, $s1 = -0.8$ and $s2 = 0.2$.

Record the long-term behaviour of the sequence. Then repeat the process with $c = -0.5$ and $c = -0.7$. When $c = -0.7$, you may find it helpful to increase N still further; try $N = 200$.

(d) Reset $s1 = -2$, $s2 = 2$ and $N = 10$. Set $c = -1.2$.

The long-term behaviour is not clear from the first ten iterations. However, the fixed-point and 2-cycle information below the graph shows that f has two repelling fixed points and an attracting 2-cycle, so the sequence may tend to this 2-cycle. To check this, increase the number of iterations; set $N = 100$.

The values in the table do indeed appear to be settling down. To see the long-term behaviour clearly on the diagram, you can arrange for only the later iterations to be plotted. The graph is set up to omit the first M iterations from the plot, so you just need to increase the value of M; try $M = 50$.

Record the long-term behaviour of the sequence. Then repeat the process with $c = -1.1$ and $c = -1.3$.

(e) Reset $N = 10$ and $M = 0$. Set $c = -1.8$.

Again, the long-term behaviour of the sequence is not clear. The fixed-point and 2-cycle information shows that although f has two fixed points and a 2-cycle, these are all repelling.

Increase the value of N to see if the behaviour of the sequence seems to settle down. Try $N = 100$, to begin with, and then try increasing N still further, say to 200, and then to 300. You should find that new construction lines appear each time you increase N, so it seems that the sequence does not settle down. It appears to be chaotic.

Notice, however, that this sequence appears to be bounded: all terms seem to lie between about -1.8 and 1.5.

Record the long-term behaviour of the sequence. Then repeat the process with $c = -1.7$ and $c = -1.9$.

Solutions are given on page 67.

In Activity 4.2 you were introduced to some ways in which you can adjust the values of variables in the Mathcad worksheet to help you find the long-term behaviour of an iteration sequence. These methods will be useful in the next activity. It is not possible to state an infallible strategy, but the list of suggestions in the box below should be a useful reference.

Identifying the long-term behaviour of an iteration sequence

The graphical iteration diagram, the table of terms, and the fixed-point and 2-cycle information can all help you to identify the long-term behaviour of an iteration sequence. Try the following suggestions. At any stage it may be helpful to rescale the diagram by adjusting the values of $s1$ and $s2$.

To explore whether the sequence tends to a fixed point

- ◇ Check that f has a fixed point, and that it is attracting (check that $|f'(a)| < 1$, where a is the fixed point).
- ◇ Increase the number N of iterations until the sequence settles down and appears to tend to the fixed point.

The sequence has 'settled down' when increases to the number of iterations do not produce any visible new construction lines (assuming that no large area of the diagram is already completely covered by construction lines!).

To explore whether the sequence tends to a 2-cycle

- ◇ Check that f has a 2-cycle, and that it is attracting (check that $|f'(a)f'(b)| < 1$, where a and b are the members of the 2-cycle).
- ◇ Increase the number of iterations until the sequence settles down. Then increase the number M of early iterations omitted from the plot until the remaining construction lines form a clear square.

To explore whether the sequence tends to a p-cycle ($p > 2$)

Increase the number of iterations until the sequence settles down. Then increase the number of early iterations omitted from the plot until only the more settled iterations are plotted. If possible, find the number of members of the cycle by counting the vertical construction lines in the diagram.

If some members of the cycle are close together, then the graphical iteration diagram may not be clear enough to allow you to count them. Rescaling parts of the diagram may help.

To explore whether the sequence is unbounded

Increase the number of iterations until the table shows that the terms become very large and positive or very large and negative, or the calculation for $f(x_n)$ gives the error message 'Encountered a floating point error.'.

To explore whether an iteration sequence is chaotic

Increase the number of iterations (to, say, 100, then 200, then 300) and check that the sequence does not seem to settle down (check that new construction lines appear on the diagram each time the number of iterations is increased).

An apparently chaotic sequence may in fact tend to a cycle of very large order, but we ignore this possibility.

From the results obtained in Activity 4.2, it appears that between the numbers -2 and 1 there may be a range of values of c for which the sequence x_n tends to a fixed point, and a range of values of c for which it tends to a 2-cycle. In the next activity you are asked to find these ranges more precisely.

Activity 4.3 Attracting fixed points and attracting 2-cycles

By testing further values of c in the Mathcad worksheet, try to determine the ranges of values of c between -2 and 1 for which the sequence x_n behaves as follows:

(a) x_n tends to a fixed point;

(b) x_n tends to a 2-cycle.

It may help to record the results of these tests with your earlier results in Figure 4.2.

Solutions are given on page 67.

You should still be working with Mathcad file 221B1-01.

In Activity 4.3 you saw a range of values of c for which the sequence x_n tends to a fixed point, and a range of values of c for which it tends to a 2-cycle. We can also state a range of values of c for which the sequence x_n tends to infinity. As c increases, the graph of $y = x^2 + c$ moves vertically upwards. For $c > 0.25$, all parts of the graph lie above the line $y = x$, and graphical iteration shows that the iteration sequence with initial term $x_0 = 0$ tends to infinity. This is illustrated by the graphical iteration diagram for $c = 0.5$ in Figure 4.3. Thus, for all c in the open interval $(0.25, \infty)$, the sequence x_n tends to infinity.

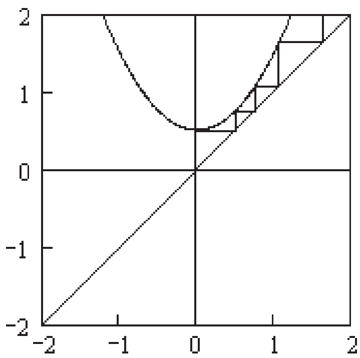


Figure 4.3 The case $c = 0.5$

In Activity 4.2, you saw that for $c = -1.3$ the sequence x_n tends to a 4-cycle, and you may have come across other values of c that give 4-cycles as you worked on Activity 4.3(b). If you wished, you could now go on to try to determine a range of values of c for which the sequence x_n tends to a 4-cycle, and then perhaps go on to look at other values of c. However, for c less than about -1.4, the way in which the behaviour of the sequence alters as c varies is very complicated. Chaotic behaviour is possible, but attracting cycles can also occur, as you will see in the next activity.

Activity 4.4 Attracting p-cycles

Set $s1 = -2$, $s2 = 2$, $N = 100$ and $M = 50$.

You should still be working with Mathcad file 221B1-01.

(a) Set c to each of the following values in turn, and in each case state the long-term behaviour of the iteration sequence x_n:

$$-1.39, -1.475, -1.575, -1.76.$$

(b) Set $c = -1.4$, and try to decide whether the iteration sequence is chaotic or tends to a p-cycle.

Solutions are given on page 67.

Comment

Once you have found a p-cycle that appears to be attracting, it is possible to check this classification of the p-cycle by calculating the corresponding gradient product; see the main text.

The final, *optional*, activity in this subsection uses the Mathcad worksheet to illustrate the fact that if the function f has a p-cycle for some value of p, then the members of the p-cycle are all fixed points of the composite function f^p.

Recall that f^p denotes the pth iterate of f, that is, the composite function $f \circ f \circ \cdots \circ f$, obtained when f is applied p times; see the main text.

When the variable p in the Mathcad worksheet is set to an integer greater than 1 (and not too large), the graph of f^p is added to the graphical iteration diagram.

Activity 4.5 p-cycles and the function f^p (Optional)

You should still be working with Mathcad file 221B1-01.

Set $s1 = -2$, $s2 = 2$, $N = 100$ and $M = 50$.

(a) Set $c = -1.2$ and $p = 2$. Observe from the graph of f^2 that the members of the 2-cycle are attracting fixed points of f^2.

(b) Repeat the process for the following values of c and p:

You have already considered these values of c, in Activities 4.2 and 4.4.

$c = -1.76$, $p = 3$; $c = -1.3$, $p = 4$;
$c = -1.475$, $p = 6$ (put $s1 = -1.8$ to obtain the graph of f^6).

Mathcad notes

The function f^p is displayed on the graph only when $p = 2, 3, 4, \ldots$. Its suppression when p is 0 or 1 is achieved by multiplying the expressions entered on the graph axes to plot the graph of f^p by $(p > 1)$. Mathcad assigns the expression $(p > 1)$ the value 0 when it is false and 1 when it is true.

Now close Mathcad file 221B1-01.

You may have wondered how we obtained the values of c given in Activity 4.4. It is not easy to find such attracting cycles by testing 'random' values of c; if you experimented using the Mathcad worksheet with values of c less than about -1.4, then you probably found that you usually obtained a sequence whose behaviour appeared to be chaotic. The next subsection shows how to *find* values of c that give cycles.

4.2 The bifurcation diagram

Recall that the sequence x_n is defined by the recurrence system in equations (4.1).

The diagram in Figure 4.4, on the next page, provides an overview of the changes in the long-term behaviour of the sequence x_n as c is varied. The diagram was generated, using Mathcad, by considering 1501 values of c, equally-spaced on the horizontal axis. For each of these values of c, the iteration sequence

$$x_0, x_1, x_2, \ldots, x_{300}$$

was calculated. The first 200 terms of this sequence were then ignored, and the remaining terms displayed on the graph, by plotting each of the points

$$(c, x_{200}), (c, x_{201}), \ldots, (c, x_{300}),$$

as a small dot.

If you pick a value of c, and imagine a vertical line drawn through the point on the horizontal axis of the diagram corresponding to that value of c, then the points where that vertical line intersects the graph give an indication of the long-term behaviour of the sequence x_n.

If there is a single point of intersection, then this suggests that the sequence tends to a fixed point. Two intersection points suggest that it tends to a 2-cycle, 3 points suggest a 3-cycle, and so on. Many points suggest chaotic behaviour. You can see from the diagram, for example, that values of c in the interval $(-1.25, -0.75)$ seem to yield sequences x_n that tend to a 2-cycle. This agrees with the solution to Activity 4.3(b).

A single intersection point occurs when the plotted terms of the sequence are very close together. Two intersection points occur when each plotted term is close to one of two values. Three or more intersection points occur in a similar way.

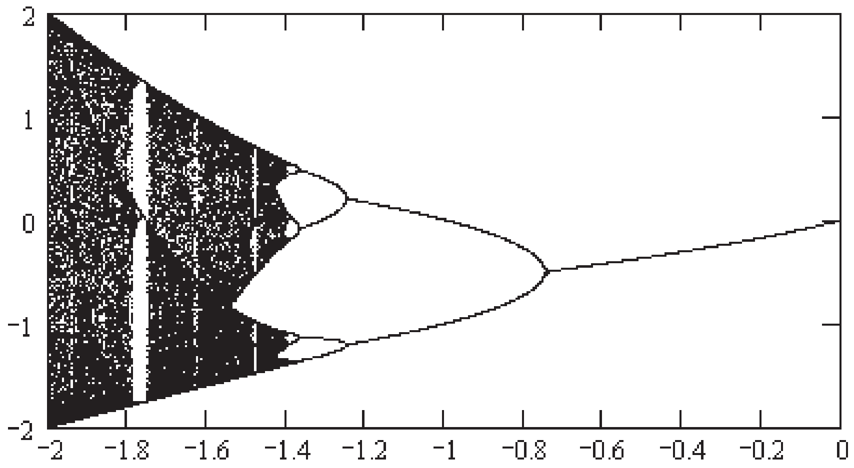


Figure 4.4 A bifurcation diagram for functions of the form $f(x) = x^2 + c$

The diagram illustrates that there are certain key values of c where the behaviour of the sequence x_n changes. For example, as c decreases through -0.75, the sequence stops tending to a fixed point, and begins to tend to a 2-cycle. Similarly, as c decreases through -1.25, the sequence stops tending to a 2-cycle, and begins to tend to a 4-cycle. At such values of c, each member of the cycle 'forks' into two cycle members, and we say that a *bifurcation* occurs. The graph in Figure 4.4 is known as a *bifurcation diagram*.

To 'bifurcate' is to fork into two.

At each bifurcation the number of members of the cycle doubles, since *every* member of the cycle 'forks' into two. For example, as c decreases from the value 0, sequences tending to fixed points give way to sequences tending to 2-cycles, which give way to sequences tending to 4-cycles, and so on. Similarly, as c decreases from the value -1.75, sequences tending to 3-cycles give way to sequences tending to 6-cycles, and so on. There are in fact infinitely many 'windows' in which the sequences tend to cycles and doubling of the number of members of the cycles occurs; in each of them the bifurcation points become progressively closer together as c decreases.

Outside these windows, sequences show bounded chaotic behaviour. For example, when $c = -1.8$ the sequence appears to be chaotic, with all terms lying between approximately -1.8 and 1.5; this confirms what you saw in Activity 4.2(e). You can see from the bifurcation diagram that as c decreases, the interval of values within which terms of the sequence lie gradually expands.

In the next activity you will use Mathcad to explore the bifurcation diagram. You will be asked to try to find a value of c that gives an iteration sequence which tends to a 5-cycle.

Activity 4.6 The bifurcation diagram

Open Mathcad file **221B1-02 Overview of iterations of f(x)=x^2+c**. This worksheet draws a bifurcation diagram for functions of the form $f(x) = x^2 + c$, with initial term $x_0 = 0$, for values of c from $C1$ to $C2$.

The bifurcation diagram in Figure 4.4 was produced by taking $V = 1500$, $N = 300$ and $M = 200$.

The bifurcation diagram is produced by taking $V + 1$ equally-spaced values of c. For each of these values of c, N iterations are carried out, and the first M of these are omitted from the plot. The worksheet is set up with $V = 500$, $N = 300$ and $M = 200$.

If you seek to change both $C1$ and $C2$, then you will find that Mathcad starts to recalculate after the first change has been made. The Comment below gives ways to deal with this problem.

You can see parts of the diagram in more detail by altering the values of the horizontal axis limits $C1$ and $C2$. You can also increase the value of V; this produces a clearer diagram, but the calculations take longer.

Try to find a value of c with an attracting 5-cycle, by proceeding as follows.

(a) First set $C2 = -1.5$, so that you can see the part of the diagram corresponding to values of c between -2 and -1.5 in more detail.

(b) Look at the part of the diagram between -1.65 and -1.6. You should see a narrow window that appears to include values of c for which the sequence x_n tends to a 5-cycle. Set $C1 = -1.65$ and $C2 = -1.6$ to confirm this observation, and choose a value of c that gives a 5-cycle.

(c) If you wish, test your value of c in Mathcad file 221B1-01.

A solution to part (b) is given on page 67.

Comment

◇ You can interrupt a Mathcad calculation by pressing `[Esc]`, the escape key, and then clicking 'OK' in the resulting option box. To resume calculation, press the `[F9]` function key, or go to the **Tools** menu, then choose **Calculate** and click on **Calculate Now**.

◇ By default, Mathcad operates in 'automatic calculation mode', but this can be inconvenient where more than one input change is to be made before recalculation is required. In order to switch to 'manual calculation mode' (which disables automatic calculation), select **Calculate** from the **Tools** menu, then click on **Automatic Calculation**. (When you do this, the word 'Auto' in the status bar, at the bottom right corner of the Mathcad window, is replaced by 'Calc F9'.)

In order to return to automatic mode, follow the same procedure.

Once in manual mode, you can calculate the results as and when you choose, either by selecting **Calculate** from the **Tools** menu and then clicking on **Calculate Now** or by pressing the `[F9]` function key.

More information about this technique, and details of how to interrupt and resume calculations, are provided in *A Guide to Mathcad*.

Mathcad notes

$x_{i,n}$ is the notation used in the worksheet for the nth term of the sequence obtained using the ith value of c, denoted by c_i.

◇ The graph is produced by plotting $x_{i,n}$ against c_i using the trace type 'points'. The subscripted variable $x_{i,n}$ is entered in the usual way. Either use the 'x_n' button on the 'Matrix' toolbar or type `[` (left square bracket), then separate the subscripts i and n with a comma. Thus you could obtain $x_{i,n}$ on the screen by typing `x[i,n`.

◇ A Mathcad graph can display about 490 000 points. If you try to plot a graph with more points than this, then an error may occur. (No graph is drawn and the graph box is highlighted in red – clicking on it reveals the error message 'Unable to plot this many points.'.)

Now close Mathcad file 221B1-02.

4.3 Other quadratic iteration sequences (Optional)

Other families of quadratic functions also show complicated behaviour when they are iterated. Consider, for example, the family of iteration sequences

$$x_0 = 0.5, \quad x_{n+1} = x_n + rx_n(1 - x_n) \quad (n = 0, 1, 2, \ldots), \tag{4.2}$$

obtained by iterating the quadratic function $f(x) = x + rx(1 - x)$, where r is a parameter. Different values of r determine different iteration sequences from this family, in the same way that different values of c determine different iteration sequences from the family in equations (4.1). Figure 4.5 shows a bifurcation diagram for this new family, with r between 1 and 3. The structure of the graph appears essentially the same as that of the bifurcation diagram in Figure 4.4, although it is reversed and distorted.

If you have studied Chapter B1 of MST121, then you may recognise that this sequence is associated with the *logistic recurrence relation*, and you will have seen the diagram in Figure 4.5 in connection with this.

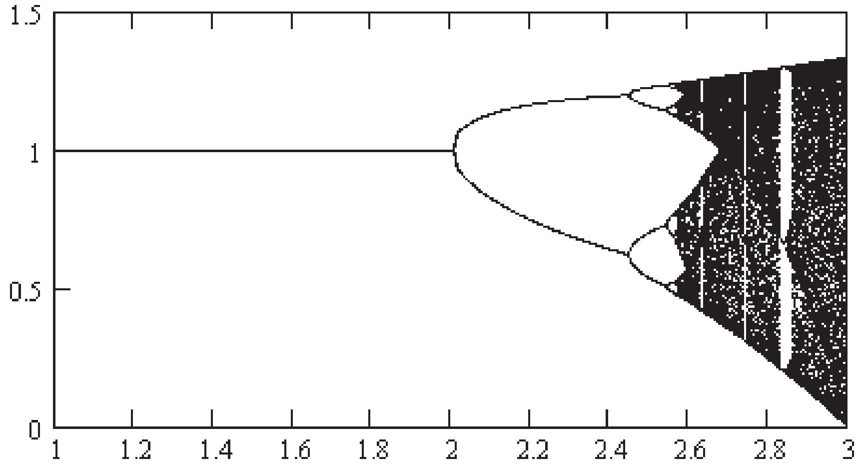


Figure 4.5 A bifurcation diagram for functions of the form $f(x) = x + rx(1 - x)$

Essentially the same bifurcation diagram occurs for various families of iteration sequences, generated both by quadratic functions and by other functions.

The similarity of the bifurcation diagrams for these two families can be explained as follows. Suppose that the sequence x_n satisfies equations (4.2). Consider the new sequence

$$x'_n = -rx_n + \tfrac{1}{2}(1 + r) \quad (n = 0, 1, 2, \ldots).$$

This sequence x'_n is related to x_n by scaling (by the factor $-r$) and shifting (by adding $\frac{1}{2}(1 + r)$). Thus both the sequences x_n and x'_n have the same *type* of long-term behaviour (for example, if x_n tends to a 3-cycle, then x'_n tends to a 3-cycle). Now, it can be shown that the new sequence x'_n satisfies

$$x'_0 = 0.5, \quad x'_{n+1} = (x'_n)^2 + c \quad (n = 0, 1, 2, \ldots), \tag{4.3}$$

where $c = \frac{1}{4}(1 - r^2)$. Equations (4.3) are of the same form as equations (4.1), except that the initial term has changed from 0 to 0.5. It can be shown that this change of initial term does not affect the long-term behaviour of these sequences. Thus the long-term behaviour seen on a vertical slice through the graph in Figure 4.5, for some particular value of r, is of the same *type* as that seen on the vertical slice through the graph in Figure 4.4 for the corresponding value $c = \frac{1}{4}(1 - r^2)$.

You can check this, if you wish, by expressing x_n in terms of x'_n and substituting for x_0, x_n and x_{n+1} in equations (4.2).

Now as the parameter r takes values increasing from 1 to 3, the parameter $c = \frac{1}{4}(1 - r^2)$ takes values decreasing from 0 to -2. Thus the patterns in Figure 4.5 occur in the reverse order to those in Figure 4.4. They are also more 'squashed together' at one end because $\frac{1}{4}(1 - r^2)$ decreases progressively faster as r increases from 1 to 3. Other distortions occur because the scaling and shifting factors relating x_n to x'_n vary as r varies.

4.4 Sequences obtained by iterating real functions

A strategy for using this file is given in the box below.

In this subsection you will use Mathcad file 221B1-03, which allows you to explore the sequences obtained by iterating *any* real function.

If you have done the computer work for Chapter A3 of MST121, then you will have met the *solve block*. It is also covered in *A Guide to Mathcad*.

If we wish to study the long-term behaviour of the sequences obtained by iterating a given real function f, then it is useful to find and classify any fixed points and 2-cycles of f. The Mathcad worksheet uses the Mathcad *solve block* to find approximate solutions to the fixed point equation and the 2-cycle equation. The solve block can be used whether or not these equations have solutions given by formulas. You have to provide Mathcad with a 'guess' for a solution; it then uses an iterative method to calculate a sequence of values that become progressively closer to a solution. Mathcad usually obtains a solution accurate to several decimal places; if it cannot find one, then it registers an error. Different initial guesses may yield different solutions.

In Block C, you will see that the gradient $f'(x)$ of the graph of a function f at $(x, f(x))$ is also represented by the notation

$$\frac{d}{dx}(f(x)),$$

called the *derivative* of f at x.

To classify the fixed points and 2-cycles of a function f, we have to be able to find the gradient of the graph of f at the fixed points, and at members of the 2-cycles. The Mathcad worksheet uses a built-in feature of Mathcad, the $\frac{d}{dx}$ operator, to find an approximate value for the gradient, here denoted by the expression $Df(x)$. You will learn much more about this topic in Block C; for now, just accept the values that Mathcad provides!

In the next activity you are asked to explore iterations of a particular real function, using the following strategy.

This strategy should be read while you are doing Activity 4.7 on the next page.

Exploring iteration sequences using Mathcad file 221B1-03

Edit the definition of the function f, on page 2, as required.

To find and classify all the fixed points of f (page 2)

1. If necessary, rescale the graph to show all the fixed points of f. (Adjust the values of the axis limits $s1$ and $s2$.)
2. Use the graph to estimate the approximate value of a fixed point of f, and set the 'guess' value x to this value. Read off the more accurate value a given for the fixed point. Use the value of the gradient $f'(a)$ to classify the fixed point.
3. Repeat the instructions in step 2 as many times as is necessary to find and classify all the fixed points of f.

The value a is indicated on the graph by a small black box at the point (a, a). If a is not the value that you wished to find, then try to set the 'guess' value x more accurately.

To find and classify all the 2-cycles of f (page 3)

1. If necessary, rescale the graph to show all the fixed points of the composite function $f \circ f$. Identify the fixed points of $f \circ f$ that are not fixed points of f (these are the points where the graph of $f \circ f$ meets the line $y = x$ but the graph of f does not). These points pair off into 2-cycles of f.
2. Use the graph to estimate the approximate value of a fixed point of $f \circ f$ that is not a fixed point of f, and set the 'guess' value x to this value. Read off the more accurate value a, and the corresponding value $b = f(a)$, for the 2-cycle a, b of f. Use the value of the gradient product $f'(a)f'(b)$ to classify the 2-cycle.
3. Repeat the instructions in step 2 as many times as is necessary to find and classify all the 2-cycles of f.

The values a and b are indicated on the graph by small black boxes at the points (a, a) and (b, b), joined by dashed black lines. If a is not the value that you wished to find, then try to set the 'guess' value x more accurately.

To identify the long-term behaviour of a sequence (page 4)

Use the suggestions given in the box on page 42 to help you to find the long-term behaviour of the iteration sequence.

The variables N, M, $s1$, $s2$ and p have the same roles here as in Subsection 4.1.

Activity 4.7 Iterating real functions

Open Mathcad file **221B1-03 Iterating real functions**. The function

$$f(x) = \frac{4x}{(1+x^2)^3}$$

is already entered for you on page 2 of the worksheet. Use the worksheet and the strategy on page 48 to help you to do the following.

(a) Find all the fixed points of f, and classify them as attracting, repelling or indifferent.

(b) Find all the 2-cycles of f, and classify them as attracting, repelling or indifferent.

(c) For each of the following initial terms x_0, find the long-term behaviour of the sequence obtained by iterating f with that initial term:

$$x_0 = -1, \quad x_0 = 1.$$

Solutions are given on page 67.

Mathcad notes

- ◇ Both the solve block (used to find a solution of an equation) and the $\frac{d}{dx}$ operator (used to calculate a gradient) use numerical methods to obtain approximations to the exact solution and gradient, respectively. The error messages 'This variable is undefined.' (for the solve block) and 'This calculation does not converge to a solution.' (for the $\frac{d}{dx}$ operator) indicate that the numerical method has failed. In the case of the solve block it is worth trying different initial guesses, though there may in fact be no solution. In the case of the $\frac{d}{dx}$ operator, you could try to find the gradient at a nearby point.
- ◇ The accuracy of the solve block is controlled by the built-in variable TOL (**Tools** menu, **Worksheet Options...**, 'Built-In Variables' tab, Convergence Tolerance (TOL)). Mathcad looks for a solution until the difference between two successive approximations is less than or equal to TOL. By default, TOL = 0.001, but for this file it is set to 0.000 001. For the $\frac{d}{dx}$ operator, the value given is usually accurate to 7 or 8 significant figures, irrespective of the value of TOL.
- ◇ The choice of the name Df for the gradient function of f in the Mathcad worksheet is a pragmatic one. The usual notation, f' ('f prime' or 'f dash'), could have been used, but the prime symbol is hard to see when placed to the right of an '`f`' in Mathcad.

The prime symbol is obtained in Mathcad by pressing the left quote key, or `[Ctrl][F7]`.

Now close Mathcad file 221B1-03.

Chapter B2, Section 5
Visualising affine transformations

Mathcad provides a convenient means by which you can see the effects of linear and affine transformations on the unit square and unit grid.

5.1 Linear transformations

In the first activity you will explore linear transformations of the plane. You will use a Mathcad worksheet that allows you to enter any 2×2 matrix $\mathbf{A}$ and see the effect of the linear transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ on the unit square and unit grid.

You can take f to be one of the basic linear transformations by setting $\mathbf{A}$ equal to one of the following matrices, expressed in 'Mathcad notation'.

This notation is used in all the Mathcad worksheets for Chapters B2 and B3.

Mathcad matrix notation for basic linear transformations

Linear transformation	**Mathcad matrix notation**
Rotation about the origin through angle θ	$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$
Reflection in the line through the origin that makes angle θ with the positive x-axis	$Q(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$
Scaling with factor a in the x-direction and factor b in the y-direction	$S(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$
Uniform scaling with factor a	$U(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$
x-shear with factor a	$X(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$
y-shear with factor a	$Y(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$

For example, the matrix of a rotation about the origin through angle $\frac{1}{2}\pi$ anticlockwise can be specified in the worksheet by entering '$R(\frac{\pi}{2})$'. Note that our Mathcad notation for rotation and reflection matrices differs slightly from the notation used in the main text. In the Mathcad notation, parameters appear in brackets (for example, $R(\frac{\pi}{2})$), whereas in the main text they appear as subscripts (for example, $\mathbf{R}_{\pi/2}$). The Mathcad notation is used not only in the Mathcad worksheets but also in this computer book.

In the first worksheet, the side of the unit square that lies along the x-axis is marked with a filled-in triangle, as shown in Figure 5.1(a), on the next page. The image of this triangle is plotted on the image square, so that you can tell which vertices of the square have been mapped to which

vertices of the image. This lets you see whether orientation has been preserved or reversed. For example, Figure 5.1(b) shows the image of the unit square under a rotation through $\frac{1}{2}\pi$ anticlockwise about the origin, and Figure 5.1(c) shows its image under reflection in the y-axis.

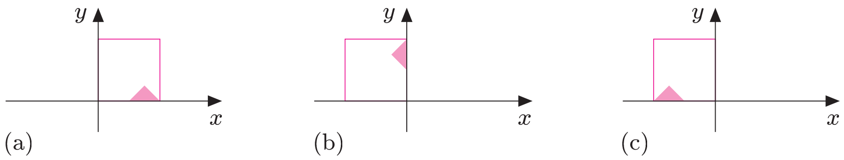


Figure 5.1 The unit square and two images

You saw in the main text that you can predict whether a linear transformation f preserves or reverses orientation by considering the determinant of its matrix, $\mathbf{A}$. The transformation f preserves orientation if $\det \mathbf{A} > 0$ and reverses orientation if $\det \mathbf{A} < 0$. If $\det \mathbf{A} = 0$, then f is a flattening, which destroys orientation. The determinant of $\mathbf{A}$ also tells you the effect of f on areas: these are scaled by the factor $|\det \mathbf{A}|$.

See Chapter B2, Section 2.

Activity 5.1 Images of the unit square and unit grid

Open Mathcad file **221B2-01 Linear transformations**. Page 2 of the worksheet describes the basic techniques for creating, editing and calculating with matrices in Mathcad. Page 3 sets up and explains the notation used in the worksheet.

These matrix techniques are used in the computer work for Chapter B2 of MST121.

Move to page 4. Here a 2×2 matrix $\mathbf{A}$ is defined, and its determinant calculated. Graphs display the unit square and unit grid, and their images under the linear transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$. You can set $\mathbf{A}$ to be a matrix representing one of the basic linear transformations, using the notation explained before this activity. Alternatively, you can set $\mathbf{A}$ to be the general 2×2 matrix $\mathbf{M}$. The page is set up with $\mathbf{A} = R(\frac{\pi}{2})$.

The determinant of $\mathbf{A}$ is denoted by $|\mathbf{A}|$ in Mathcad.

The graphs can be rescaled by changing the value of s, which is defined near the top of the page; both axes of both graphs are scaled from $-s$ to s.

The edges of the graph box may be obscured in places by the unit grid.

Set $\mathbf{A}$ to be each of the following matrices in turn. In each case, use the value of the determinant of $\mathbf{A}$ to predict whether the transformation preserves, reverses or destroys orientation, and whether it decreases, preserves or increases area, and try to confirm your predictions by looking carefully at the effect of the transformation on the unit square.

(a) Rotations: $R(\frac{\pi}{4})$, $R(-\frac{\pi}{4})$.

(b) Reflections: $Q(\frac{\pi}{4})$, $Q(\frac{3\pi}{4})$.

(c) Scalings: $S(2,1)$, $S(0.5,1)$, $S(-1,1)$, $S(-1,-3)$.

(d) Uniform scalings: $U(2)$, $U(0.5)$, $U(-1)$, $U(-0.5)$.

(e) x-shears: $X(2)$, $X(1)$, $X(0)$, $X(-1)$, $X(-2)$.

(f) y-shears: $Y(2)$, $Y(-1)$.

(g) A flattening: $\begin{pmatrix} 4 & -6 \\ -2 & 3 \end{pmatrix}$.

(h) Non-basic linear transformations: $\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$, $\begin{pmatrix} 0.4 & -1 \\ -0.4 & -0.4 \end{pmatrix}$.

For parts (g) and (h), define $\mathbf{A} := \mathbf{M}$. The matrix in part (g) is already entered for you; edit $\mathbf{M}$ to obtain the matrices in part (h).

Comment

- Parts (a) to (g) illustrate the effects of basic linear transformations on orientation and area, which can be summarised as follows.

  Remember that a uniform scaling $U(a)$ is a scaling, with both factors equal to a.

  Orientation-preserving transformations: rotations, shears, scalings with factors a, b of the same sign.

  Orientation-reversing transformations: reflections, scalings with factors a, b of opposite signs.

  Area-preserving transformations: rotations, reflections, shears, scalings with factors a, b where $|ab| = 1$.

  Area-decreasing transformations: scalings with factors a, b where $|ab| < 1$, flattenings.

  Area-increasing transformations: scalings with factors a, b where $|ab| > 1$.

  Flattenings destroy orientation and decrease area to zero.
- The linear transformation represented by the first matrix in part (h) preserves orientation and increases area; the second reverses orientation and decreases area.

We now consider general linear transformations. For example, Figure 5.2 shows the effect on the unit square and unit grid of the linear transformation m represented by the matrix $\mathbf{M} = \begin{pmatrix} 0 & -2 \\ 2 & 2 \end{pmatrix}$.

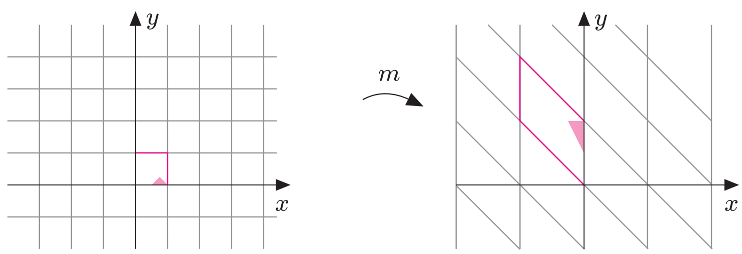


Figure 5.2 Image under the linear transformation m

This linear transformation is in fact the composite of a uniform scaling with factor 2, followed by an x-shear with factor 1, followed by a rotation through $\frac{1}{2}\pi$, as we now check using matrix multiplication.

See Chapter B2, Section 3. Remember that $g \circ f$ means 'first apply f, then apply g'.

Recall that if f and g are linear transformations represented by matrices $\mathbf{A}$ and $\mathbf{B}$ respectively, then the composite linear transformation $g \circ f$ is represented by the product matrix $\mathbf{BA}$. It follows that if f, g and h are linear transformations of the plane represented by matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ respectively, then the composite linear transformation $h \circ g \circ f$ is represented by the matrix $\mathbf{CBA}$.

If we take $\mathbf{A}$ to be the matrix representing a uniform scaling with factor 2, $\mathbf{B}$ to be the matrix representing an x-shear with factor 1, and $\mathbf{C}$ to be the matrix representing a rotation through $\frac{1}{2}\pi$, then

$$\mathbf{CBA} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 2 \end{pmatrix} = \mathbf{M},$$

as required.

Any linear transformation m of the plane can be expressed as a composite of at most three basic linear transformations. This shows that every linear transformation of the plane has a fairly simple geometrical interpretation. In particular, if m is not a flattening, then it can be expressed as the composite of a scaling, followed by an x-shear, followed by a rotation.

Some of the basic linear transformations in this 'decomposition' of m may be the identity transformation, in which case m is a composite of two basic linear transformations, or is itself a basic linear transformation.

The uniform scaling with factor 1, the x-shear with factor 0 and the rotation through angle 0 are all equal to the identity transformation.

To express a given linear transformation that is not a flattening as a composite of a scaling, followed by an x-shear, followed by a rotation, we consider its effect on the unit square. Figure 5.3(a) shows the unit square and Figure 5.3(d) shows its image under a typical linear transformation m. The image is a parallelogram with one vertex at the origin, which we call the *target parallelogram*. We use the word *base* to describe the side of the unit square that lies along the x-axis and the image of this side on the target parallelogram. These base sides are marked with filled-in triangles. Figure 5.3(b) and (c) illustrate stages in finding the decomposition, with the corresponding base sides marked similarly.

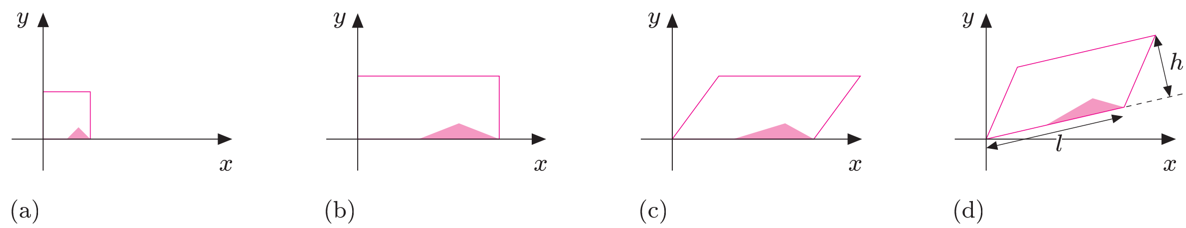


Figure 5.3 Successive images of the unit square

The process of finding the basic linear transformations that constitute the decomposition of m is described below.

Decomposition of a linear transformation m

Suppose that the target parallelogram under m has a base of length l, and height h, as shown in Figure 5.3(d). The three basic linear transformations for the decomposition can be chosen as follows.

1. First choose the scaling with matrix $S(\pm l, h)$, where the $+$ sign is used if m preserves orientation and the $-$ sign otherwise. This maps the unit square to a rectangle with base of length l and height h; see Figure 5.3(b).
2. Next take the x-shear that maps the rectangle onto a parallogram that can be rotated onto the target parallelogram; see Figure 5.3(c).
3. Finally, take the rotation about the origin that maps the parallelogram onto the target parallelogram in Figure 5.3(d).

This rectangle has the same orientation and area as the target parallelogram.

In step 2, you can find the x-shear required by determining the signed distance, d say, that the x-shear moves a point on the side of the rectangle opposite the base. The shear factor is then d/h.

Here d is positive if the point moves to the right and negative if it moves to the left.

In the next activity you will use a page of the Mathcad worksheet that is designed to help you carry out this decomposition process for a given linear transformation. Parts (a) and (b) of the activity provide examples, and in part (c) you are asked to work through the process yourself.

Activity 5.2 Composite linear transformations

You should still be working with Mathcad file 221B2-01.

Move to page 5 of the worksheet. Near the top of the page a matrix $\mathbf{M}$ is defined, and the value of its determinant $|\mathbf{M}|$ is displayed. Graphs display the unit square and unit grid, and their images under the linear transformation $m(\mathbf{x}) = \mathbf{Mx}$. The bases of the unit square and the target parallelogram (shown in blue) are marked with filled-in triangles.

Beneath this, the page is set up so that you can define the matrix $\mathbf{A}$ of a scaling, the matrix $\mathbf{B}$ of an x-shear, and the matrix $\mathbf{C}$ of a rotation. The determinants of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{CBA}$ are all displayed. Initially, we have

Note that each of the matrices $S(1,1)$, $X(0)$ and $R(0)$ represents the identity transformation.

$$\mathbf{A} = S(1,1), \qquad \mathbf{B} = X(0), \qquad \mathbf{C} = R(0). \tag{5.1}$$

The linear transformations represented by $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are defined as f, g and h, respectively. The images of the unit square and unit grid under f, $g \circ f$ and $h \circ g \circ f$ are displayed in successive graphs. Each of these graphs also shows the target parallelogram, in dashed blue lines. All the graphs can be rescaled by changing the value of s, defined near the top of the page.

(a) The page is set up with

The decomposition of this matrix $\mathbf{M}$ was discussed earlier in the subsection.

$$\mathbf{M} = \begin{pmatrix} 0 & -2 \\ 2 & 2 \end{pmatrix}.$$

In this case $|\mathbf{M}|$ is positive, so m preserves orientation.

Set each of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as shown below, in turn, and as you do so check that they achieve steps 1, 2 and 3, respectively, of the decomposition process on page 53.

$$\mathbf{A} = S(2,2) \qquad \mathbf{B} = X(1) \qquad \mathbf{C} = R(\tfrac{\pi}{2})$$

(b) Now reset each of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as in equations (5.1). Then edit the entries of the matrix $\mathbf{M}$ to give

$$\mathbf{M} = \begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}.$$

In this case $|\mathbf{M}|$ is negative, so m reverses orientation.

Set each of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as shown below, in turn, and as you do so check that they achieve steps 1, 2 and 3, respectively, of the decomposition process on page 53.

$$\mathbf{A} = S(-1,2) \qquad \mathbf{B} = X(-\tfrac{1}{2}) \qquad \mathbf{C} = R(0)$$

(c) In each of parts (i) to (iv) below, first reset each of the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as in equations (5.1), and set $\mathbf{M}$ to the given matrix. Then follow the decomposition process to express m as a composite of basic linear transformations. Hence express each given matrix as a product of matrices of basic linear transformations.

For each of these matrices, the parameters of the required scaling and shear are small integers or simple fractions, and the parameter of the required rotation is an integer multiple of $\pi/2$.

(i) $\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ (iv) $\begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix}$

Solutions to part (c) are given on page 68.

Now close Mathcad file 221B2-01.

5.2 *Affine transformations*

In the next activity you will use Mathcad to visualise the effect of some affine transformations of the plane. You will be given the images of the unit square and unit grid under various 'mystery' affine transformations, and your task is to find the affine transformations that produce these images.

Activity 5.3 Exploring affine transformations

Open Mathcad file **221B2-02 Affine transformations**. Page 2 of the worksheet defines the matrices representing the basic linear transformations, using the Mathcad notation.

Move to page 3. The unit square and unit grid, and their images under a 'mystery' affine transformation, are displayed near the top of the page. The page can be set up with any one of four different mystery transformations, and you can choose which one of these is applied by setting the variable T to 1, 2, 3 or 4. All of the graphs can be rescaled by changing the value of s, defined near the top of the page.

You can define your own affine transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ by editing the definitions of the matrix $\mathbf{A}$ and vector $\mathbf{a}$ near the middle of the page. The unit square and unit grid, and their images under f, are displayed immediately below these definitions. The graph that displays the images under f also shows the image of the unit square under the mystery affine transformation, in dashed blue lines; this is the target.

Set T to each of 1, 2, 3 and 4 in turn. In each case, try to identify the mystery affine transformation, and verify your answer by setting $\mathbf{A}$ and $\mathbf{a}$ and checking that you hit the target.

In each of the given examples, $\mathbf{A}$ is the matrix of a basic linear transformation, and the vector $\mathbf{a}$ has integer components.

Solutions are given on page 68.

Comment

You may wish to use the Mathcad page to see the effects of other affine transformations on the unit square and unit grid. If you set $T = 0$, then no target will appear on the graphs.

Mathcad notes

The definitions of the four mystery affine transformations are hidden in the columns of matrices to the right of page 3 of the worksheet. The area beyond the right-hand margin of a Mathcad page (which is marked by a solid vertical line) can be used just like the rest of the worksheet. It is divided into further pages, where you can place mathematical expressions, text, graphs and pictures. Some of the other Mathcad worksheets in MS221 also have material off the page to the right.

You can view the pages in this area by using the horizontal scroll bar to move to the right. When printing a wide worksheet, you can choose whether or not the content to the right of the left-hand pages is to be printed. To do this, select **Page Setup...** from the **File** menu. Then, in the 'Page Setup' option box, tick or untick the check box for 'Print single page width', as required. If there is a tick here, then just the left-hand pages of the worksheet will be printed when **Print...** is selected subsequently; this is the default for MS221 worksheets. Otherwise, the whole width of the worksheet will be printed.

However, it is not necessary to view this area in order to carry out the activities based on the worksheet, nor to understand the principles behind them.

See Chapter B2, Section 4.

Given any three points, in any order, there is a unique affine transformation f that maps the points $(0,0)$, $(1,0)$, $(0,1)$ to these points, respectively. The main text gave a method for finding f. The next, *optional,* activity allows you to explore the effect of affine transformations on the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$, which we call the *unit triangle*. This lets you check visually, using Mathcad, that an affine transformation found using the given method does indeed map the three points $(0,0)$, $(1,0)$, $(0,1)$ to the required image points.

Activity 5.4 Applying affine transformations to the unit triangle (Optional)

You should still be working with Mathcad file 221B2-02.

Move to page 4 of the worksheet. Here you can define an affine transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ by setting the matrix $\mathbf{A}$ and vector $\mathbf{a}$ as described in the worksheet. Graphs display the unit grid and unit triangle, and their images under f. The coordinates of the vertices of the image triangle, and its area, are shown at the bottom of the page.

Use the Mathcad page to verify the correctness of the affine transformations found in Example 4.1 and Activity 4.2 in the main text.

Mathcad notes

The '$|x|$' button (available on the 'Calculator' toolbar), whose keyboard alternative is `[Shift]\` (shift and backslash), performs several roles in Mathcad. It gives the determinant of a matrix, the modulus (absolute value) of a real number, the magnitude of a vector, and the modulus of a complex number.

You will study complex numbers later in MS221.

The modulus of the determinant of a matrix $\mathbf{A}$ (written as $|\det \mathbf{A}|$ in the main text) can be obtained by two applications of '$|x|$', and appears as $||\mathbf{A}||$.

There is also a '$|x|$' button on the 'Matrix' toolbar. However, this can *only* be used for finding the determinant of a square matrix; it does not work for the other purposes described above. If this '$|x|$' button is applied twice to the matrix $\mathbf{A}$, to obtain $||\mathbf{A}||$, and then evaluation is sought with `=`, no result is obtained. The inner $|\mathbf{A}|$ is highlighted in red, since it is a scalar number rather than a matrix. Clicking on the red $|\mathbf{A}|$ reveals the error message 'This value must be a matrix of scalar elements.'.

Now close Mathcad file 221B2-02.

Chapter B3, Section 5
Iterating linear transformations with the computer

In this section you will use Mathcad to explore sequences of points in the plane obtained by iterating linear transformations. You will also see a few examples of sequences obtained by iterating affine transformations. The section ends with an *optional* subsection in which you can see some visually interesting plots obtained by an iteration process involving more than one affine transformation.

The Mathcad notation for matrices of basic linear transformations used in the computer work for Chapter B2 will be used again in this section, both in the text and in the Mathcad files.

This notation is given on page 50.

5.1 Iterating linear transformations

This subsection is concerned with sequences of points in the plane obtained by iterating linear transformations. In the main text, you saw some examples of such sequences where the matrix representing the linear transformation has two distinct non-zero eigenvalues, the case of a so-called *generalised scaling*. These have the following properties.

See Chapter B3, Section 4.

Iteration properties of generalised scalings

Let the linear transformation f be represented by a 2×2 matrix $\mathbf{A}$ that has two distinct non-zero eigenvalues k_1 and k_2, with corresponding eigenlines ℓ_1 and ℓ_2. Let (x_0, y_0) be a point of $\mathbb{R}^2$ and let (x_n, y_n) be an iteration sequence generated by $\mathbf{A}$, with initial point (x_0, y_0).

(a) (i) If $k_1 > 0$, then (x_n, y_n) all lie on the same side of ℓ_2 as (x_0, y_0).
  (ii) If $k_1 < 0$, then (x_n, y_n) alternate between opposite sides of ℓ_2.

(b) (i) If $\max\{|k_1|, |k_2|\} > 1$, then the sequence moves away from $(0, 0)$.
  (ii) If $\max\{|k_1|, |k_2|\} < 1$, then the sequence moves towards $(0, 0)$.

(c) If $|k_1| > |k_2|$ and (x_0, y_0) does not lie on an eigenline, then

$$\frac{y_n}{x_n} \to m \text{ as } n \to \infty,$$

where m is the gradient of ℓ_1.

Property (c) states that if k_1 is the 'dominant eigenvalue', then (x_n, y_n) tends in the direction of the 'dominant eigenline' $y = mx$.

The first activity involves sequences obtained by iterating non-uniform scalings. These are the 'simplest' generalised scalings. The scaling with factors a and b, where $a \neq b$, has eigenvalues a and b. The corresponding eigenlines are $y = 0$ (the x-axis) and $x = 0$ (the y-axis), respectively.

For example, Figure 5.1, overleaf, shows the first few points of the iteration sequence generated by $\mathbf{A} = S(1.5, 1)$ and the initial point $(2, 1)$. It illustrates behaviour that can be predicted using the iteration properties of generalised scalings given in the box above. Since both eigenvalues are positive, all points of the sequence lie on the same side of the eigenline $y = 0$, and on the same side of the eigenline $x = 0$, as the initial point. Since $\max\{|1.5|, |1|\} > 1$, the sequence moves away from $(0, 0)$.

Since $|1.5| > |1|$, the dominant eigenvalue is 1.5, the dominant eigenline is $y = 0$, and the sequence tends in the direction of this line.

It follows from property (b) of generalised scalings (see Chapter B3, Section 3) that this sequence remains the same distance from the line $y = 0$ and moves away from the line $x = 0$.

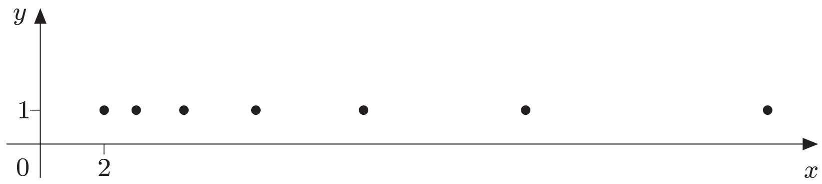


Figure 5.1 Iteration sequence generated by a non-uniform scaling

In Activity 5.1 you will see sequences obtained by iterating various non-uniform scalings.

Activity 5.1 Non-uniform scalings

Open Mathcad file **221B3-01 Iterating linear transformations**. Page 2 of the worksheet defines the matrices representing basic linear transformations, using the Mathcad notation.

Move to page 3. Here a 2×2 matrix $\mathbf{A}$ and an initial point (x_0, y_0) are defined; the initial point is defined as the vector $\mathbf{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

In this computer book,
$$\mathbf{x}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$
for $n = 0, 1, 2, \ldots$.

A graph displays the first point, as a black box symbol, and N subsequent points, as magenta box symbols, of the iteration sequence $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$, where $f(\mathbf{x}) = \mathbf{Ax}$. A solid magenta box distinguishes $\mathbf{x}_N$, the final point calculated. Mathcad also calculates and plots the eigenlines of the matrix $\mathbf{A}$ (if there are any); if there are two eigenlines and one is dominant, then the dominant one is shown in red, and the other in blue. You can rescale the graph by altering the value of s.

In this activity, you will only set $\mathbf{A}$ to be a scaling.

You can set $\mathbf{A}$ to be a matrix representing one of the basic linear transformations (or a product of such matrices). Alternatively, you can set $\mathbf{A}$ to be the 2×2 matrix $\mathbf{M}$, whose entries you can edit.

Figure 5.1 illustrates the sequence generated by this choice of $\mathbf{A}$ and (x_0, y_0).

The page is set up with $\mathbf{A} = S(1.5, 1)$, $(x_0, y_0) = (2, 1)$, $N = 1$ and $s = 25$.

(a) Set N to 2, 3 and 4 in turn, so the third, fourth and fifth points of the sequence appear on the graph. (Making the points appear one at a time shows the order in which they appear in the sequence.)

When you do this, only a few points of the sequence appear on the graph, because the subsequent points lie outside the graph box. You can rescale the graph to see more points, if you wish.

(b) Set N to 10, 20 and 30 in turn, and describe the effect on the value of the ratio y_N/x_N, which is displayed to the left of the graph. Explain how you could have predicted this.

(c) Set the initial point (x_0, y_0) to $(-2, 1)$. Observe that the iteration sequence now tends in the negative direction of the x-axis.

Experiment with a few other initial points of your own. In each case, predict the behaviour of the sequence and check your prediction.

(d) Reset the initial point to $(2, 1)$, and ensure that $N = 30$ and $s = 25$.

Set $\mathbf{A}$ to each of the non-uniform scaling matrices below in turn, and in each case check that the behaviour of the sequence is as predicted by the iteration properties of generalised scalings.

For the last of these matrices, rescale the graph by setting $s = 3$.

$$S(1.5, 1.2),\ S(1.5, 0.8),\ S(1.5, -1.2),\ S(1.5, -0.8),\ S(0.8, -0.8).$$

Solutions to parts (b) and (c) are given on page 68.

Mathcad notes

The solid box that marks the point (x_N, y_N) on the graph is obtained by plotting *three* traces, using a box, a '$\times$' and a '+'. (The traces are plotted using weight 'p', which draws the smallest symbols possible.) When superimposed on the screen, these three symbols give the appearance of a solid box. The individual symbols may become visible if printed.

In the next activity you will consider iteration sequences produced by generalised scalings for which the eigenlines are different from the axes. You have already seen some examples of such sequences in the main text. For example, you considered the linear transformation with matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix},$$

See Chapter B3, Section 3.

which has eigenvalues 4 and -1, with corresponding eigenlines $y = \frac{3}{2}x$ and $y = -x$, respectively. By the iteration properties of generalised scalings, the long-term behaviour of iteration sequences whose initial point is not on the eigenlines can be described in this case as follows.

The points of such an iteration sequence alternate between opposite sides of the eigenline $y = \frac{3}{2}x$ and lie on the same side of the eigenline $y = -x$ as the initial point. The sequence moves away from $(0, 0)$ and tends in the direction of the dominant eigenline $y = \frac{3}{2}x$.

Figure 5.2 shows the first few points of the iteration sequence with initial point $(2, 1)$.

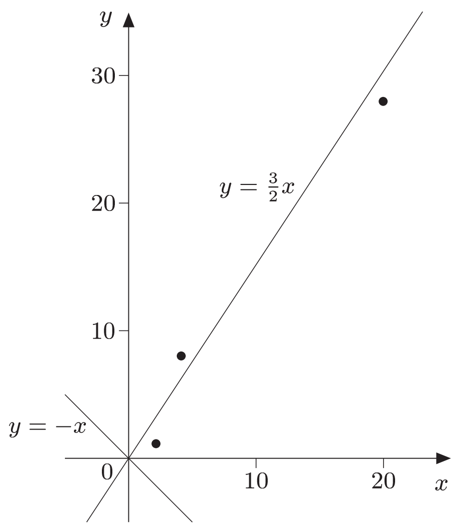


Figure 5.2 Iteration sequence generated by a generalised scaling

The next activity involves examples of a similar kind.

Activity 5.2 Linear transformations with two distinct eigenvalues

You should still be working with Mathcad file 221B3-01.

Ensure that the number of iterations, initial point and graph scale on page 3 of the worksheet are set as follows:

$$N = 30, \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad s = 100.$$

In each of parts (a) to (d) below, a matrix $\mathbf{A}$ is given, together with its eigenvalues and eigenlines. In each case, use this information together with the iteration properties of generalised scalings to predict the behaviour of the iteration sequence $\mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0$ with initial point $(2, 1)$, giving a description along the lines of the example discussed before this activity. Then confirm your answer by setting $\mathbf{A}$ appropriately in the Mathcad page and observing the graph.

In part (a), define $\mathbf{A} := \mathbf{M}$; the matrix $\mathbf{M}$ is already entered for you. For the matrices in parts (b) to (d), edit the entries of $\mathbf{M}$.

For some of these matrices, you may find it helpful to set N to 1, 2, 3 and so on, to see the order in which the points appear.

(a) $\begin{pmatrix} -2 & 1 \\ 4 & 1 \end{pmatrix}$
has eigenvalues 2 and -3, with eigenlines $y = 4x$ and $y = -x$, respectively.

(b) $\begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}$
has eigenvalues 4 and 2, with eigenlines $y = x$ and $y = \frac{1}{3}x$, respectively.

For parts (c) and (d), set $s = 3$.

(c) $\begin{pmatrix} 0.9 & 0.1 \\ -0.6 & 0.4 \end{pmatrix}$
has eigenvalues 0.7 and 0.6, with eigenlines $y = -2x$ and $y = -3x$, respectively.

(d) $\begin{pmatrix} -0.6 & 0.5 \\ 0.9 & 0.6 \end{pmatrix}$
has eigenvalues 0.9 and -0.9, with eigenlines $y = 3x$ and $y = -\frac{3}{5}x$, respectively.

Solutions are given on page 68.

In the next activity you will see some examples of sequences obtained by iterating linear transformations whose matrices have no eigenvalues. (Recall that when we say in MS221 that a matrix has no eigenvalues, we mean that it has no *real* eigenvalues. It will have *complex* eigenvalues, but we are not concerned with those.) Such sequences show a wide variety of behaviour, and you are not expected to explore them in detail.

Activity 5.3 Linear transformations with no eigenvalues

Ensure that the number of iterations, initial point and graph scale on page 3 of the worksheet are set as follows:

You should still be working with Mathcad file 221B3-01.

$$N = 200, \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad s = 5.$$

The matrices listed in parts (a) to (d) below have no eigenvalues; nor therefore do they have any eigenvectors. For each matrix, use your geometric knowledge of basic linear transformations to try to predict the general 'shape' of the iteration sequence produced by the matrix. (This is not easy for the matrices in parts (c) and (d)!) Then confirm your answer by setting $\mathbf{A}$ appropriately in the Mathcad page.

For some of these matrices, you may find it helpful to set N to 1, 2, 3 and so on, to see the order in which the points appear.

(a) Rotations:

$$R(\tfrac{\pi}{6}), \quad R(\tfrac{\pi}{3}), \quad R(1).$$

(b) Rotations composed with uniform scalings:

$$U(0.9)\,R(\tfrac{\pi}{12}), \quad U(-0.9)\,R(\tfrac{\pi}{12}), \quad U(1.01)\,R(\tfrac{\pi}{12}).$$

When setting $\mathbf{A}$ to a product of two matrices, remember to type $*$ (or use the '$\times$' button on the 'Calculator' toolbar) to indicate the multiplication.

(c) A rotation composed with a non-uniform scaling:

$$S(0.8, 1.2)\,R(\tfrac{\pi}{12}).$$

(d) Rotations composed with shears:

$$X(0.2)\,R(\tfrac{\pi}{12}), \quad X(-0.6)\,R(\tfrac{\pi}{12}).$$

Solutions are given on page 69.

Comment

No matrix of the types in parts (a) and (b), representing rotations, or composites of rotations and uniform scalings, has any eigenvalues (except where the angle of rotation is an integer multiple of π). However, some matrices of the types in parts (c) and (d), representing composites of rotations and non-uniform scalings, or composites of rotations and shears, can have two eigenvalues. For example, $S(0.8, 1.4)\,R(\frac{\pi}{12})$ and $X(0.3)\,R(\frac{\pi}{12})$ both have two eigenvalues.

In the final, *optional*, activity in this subsection, you can see a few examples of sequences obtained by iterating linear transformations whose matrices have just one eigenvalue. Again, such sequences show a wide variety of behaviour, but it is worth looking at one particular aspect of this behaviour. You have seen that if $\mathbf{A}$ is a matrix with *two* eigenvalues of different magnitudes, and the initial point $\mathbf{x}_0$ does not lie on an eigenline of $\mathbf{A}$, then

$$\frac{y_n}{x_n} \to m \text{ as } n \to \infty,$$

where (x_n, y_n) is the $(n+1)$th point of the sequence $\mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0$ and m is the gradient of the dominant eigenline. In the next activity you are invited to explore whether a similar property holds for 2×2 matrices with just one eigenvalue.

Activity 5.4 Linear transformations with one eigenvalue (Optional)

You should still be working with Mathcad file 221B3-01.

Ensure that the initial point and graph scale on page 3 of the worksheet are set as follows:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad s = 50.$$

The gradient of the eigenline is the y-component of the eigenvector with x-component equal to 1.

Each of the six matrices in parts (a) and (b) below has just one eigenvalue and just one eigenline. In each case set $N = 50$, set $\mathbf{A}$ to the given matrix, and note the gradient of the eigenline. Then increase N to 500, and check whether the value of the ratio y_N/x_N seems to be approaching the gradient of the eigenline.

For some of these matrices, you will need to rescale the graph to obtain a clear picture of the iteration sequence.

(a) (i) $X(4)$ (ii) $U(1.5)\,X(4)$ (iii) $U(0.5)\,X(4)$

In part (b), define $\mathbf{A} := \mathbf{M}$ and edit the entries of $\mathbf{M}$.

(b) (i) $\begin{pmatrix} -3 & 1 \\ -4 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 0.5 & -1 \\ 1 & 2.5 \end{pmatrix}$ (iii) $\begin{pmatrix} -1.4 & 0.2 \\ -0.2 & -1 \end{pmatrix}$

Solutions are given on page 69.

Comment

Most 2×2 matrices with just one eigenvalue have just one eigenline. The only exceptions are the matrices of uniform scalings and the zero matrix; for these matrices, *every* line through the origin is an eigenline.

One or both of the transformations forming the composite may be the identity transformation.

It can be shown that every linear transformation of the plane whose matrix has only one eigenvalue is a composite of a shear and a uniform scaling. If the matrix has only one eigenline, then the shear is parallel to this eigenline.

Mathcad notes

In part (a)(iii), you may have noticed that for both $N = 50$ and $N = 500$ Mathcad displays the last point calculated as $(x_N, y_N) = (0, 0)$ but gives a non-zero value for the ratio y_N/x_N, despite the fact that Mathcad (by default) evaluates 0/0 as 0. The reason for this is that, while Mathcad always retains values up to 17 significant figures for calculation purposes, in this worksheet numbers of magnitude less than 10^{-10} are displayed as zero on the screen. So Mathcad displays the values of $x_{50} = 1.812 \times 10^{-13}$ and $y_{50} = 8.882 \times 10^{-16}$ as zero, while displaying the calculated value of y_{50}/x_{50} as 4.902×10^{-3}.

However, for results already present in the worksheet this change must be made on an individual basis, since the worksheet-level change is not retrospective.

If you wish to display such small values, then click in an empty space in the worksheet to obtain the red cross cursor, and select **Result...** from the **Format** menu. In the 'Result Format' option box, choose the 'Tolerance' tab, and increase the value of 'Zero threshold'. For example, increasing this value from 10 to 100 will ensure that only numbers of magnitude less than 10^{-100} are displayed as zero.

Now close Mathcad file 221B3-01.

5.2 Iterating affine transformations

In Subsection 5.1 we looked at sequences of points in the plane obtained by iterating linear transformations. In the next activity you will see a few examples of iteration sequences obtained by iterating affine transformations. You saw in Chapter B2 that an affine transformation of the plane is a function of the form

$$\begin{aligned} f\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \mathbf{x} &\longmapsto \mathbf{A}\mathbf{x} + \mathbf{a}, \end{aligned}$$

where $\mathbf{A}$ is a 2×2 matrix and $\mathbf{a}$ is a vector with two components.

We generate an iteration sequence using an affine transformation f in a similar way to the other iteration sequences that you have seen: we choose an initial point $\mathbf{x}_0$ and repeatedly use the recurrence relation

$$\mathbf{x}_{n+1} = f(\mathbf{x}_n) \quad (n = 0, 1, 2, \ldots).$$

Activity 5.5 Affine transformations

Open Mathcad file **221B3-02 Iterating affine transformations**. Page 2 of the worksheet defines the matrices representing basic linear transformations, using the Mathcad notation.

Move to page 3. This is similar to page 3 of the worksheet for Mathcad file 221B3-01, but it is set up to allow you to define a vector $\mathbf{a}$ as well as a matrix $\mathbf{A}$, and the function iterated is $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$. The initial point (x_0, y_0) is set to $(2, 1)$, and the number N of iterations is set to 10.

(a) The matrix $\mathbf{A}$ is set to the rotation matrix $R(\frac{\pi}{6})$, and the vector $\mathbf{a}$ is set to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so f is initially a linear transformation.

Set $N = 20$ and describe the effect on the iteration sequence.

Then set $\mathbf{a}$ to each of the following vectors in turn and describe the effect on the iteration sequence:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(b) Set $\mathbf{A} = U(0.9)\,R(\frac{\pi}{6})$, which represents a rotation composed with a uniform scaling, and set $\mathbf{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

If you are not sure about the order in which the points appear in a sequence, then set N to 1, 2, 3 and so on, in turn, to find out.

Set $N = 40$ and describe the effect on the iteration sequence.

Then set $\mathbf{a}$ to each of the vectors in part (a) and describe the effect on the iteration sequence.

Solutions are given on page 69.

Now close Mathcad file 221B3-02.

It can be shown that if the 2×2 matrix $\mathbf{A}$ represents an anticlockwise rotation through an angle θ about the origin, then each affine transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ is an anticlockwise rotation through the angle θ about some point whose coordinates depend on θ and $\mathbf{a}$. Thus if $\mathbf{A}$ represents a rotation, then each affine transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ produces iteration sequences whose points lie on circles.

You saw in Chapter B2, Section 4, how to find $\mathbf{a}$ so that f is a rotation about a given point.

Similarly, if $\mathbf{A}$ represents a composite of a rotation and a uniform scaling, then each affine transformation $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a}$ produces iteration sequences that spiral about a point whose coordinates depend on θ and $\mathbf{a}$.

5.3 Iterated function systems (Optional)

Do not be tempted to spend too long on this subsection!

In Subsection 5.2, you looked at some sequences of points in the plane obtained by iterating affine transformations. In this subsection, you will see some sequences obtained in a similar way, but using more than one affine transformation. A collection of transformations used to generate an iteration sequence in this way is known as an *iterated function system*. Such systems can give surprising results!

See Activity 5.7, for example.

For example, consider the two affine transformations

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{a} \quad \text{and} \quad g(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{a},$$

$$\text{where } \mathbf{A} = U(0.6)\,R(0.5) \quad \text{and} \quad \mathbf{a} = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}.$$

The angle of rotation is 0.5 radians (about 29°) and the scaling factor is 0.6.

The matrix $\mathbf{A}$ represents a composite of a rotation and a uniform scaling. Both f and g produce spiral sequences if iterated on their own, the spirals centred on points other than the origin, since $\mathbf{a}$ is not the position vector of the origin. The sequences spiral inwards towards their centres, because the scaling factor has magnitude less than 1.

We can produce an iteration sequence using *both* f and g by starting with an initial point, and choosing in some way between f and g at each iteration. For example, we could simply choose either f or g at random. Figure 5.3 shows a plot of the first 50 000 points of a sequence obtained by iterating f and g in this way; the initial point is the origin.

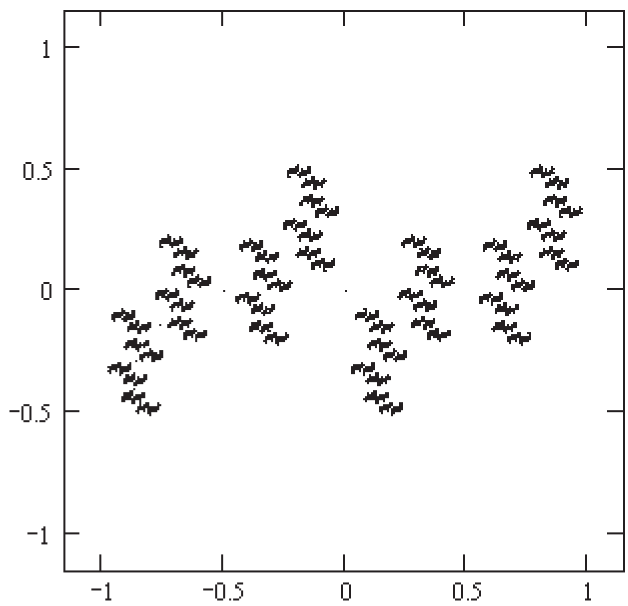


Figure 5.3 Sequence generated by an iterated function system

As you can see, the sequence produces a plot of a set that is of great complexity yet displays the property of *self-similarity*; that is, small parts of the set are 'similar' to large parts. Moreover, the set has an interesting 'natural' appearance, perhaps resembling some kind of sea life!

Complicated self-similar sets are often called *fractals*.

In the next activity you are invited to use a Mathcad page to see a variety of plots obtained in a similar way. The page allows you to define two affine transformations f and g, and an initial point $\mathbf{x}_0$. It generates and plots a sequence $\mathbf{x}_n$ using the following method. At each iteration, Mathcad generates a random number p_n between 0 and 1, and the next point of the sequence is calculated using the recurrence relation

$$\mathbf{x}_{n+1} = \begin{cases} f(\mathbf{x}_n), & \text{if } p_n < P, \\ g(\mathbf{x}_n), & \text{if } p_n \geq P, \end{cases}$$

where P is a number defined in the worksheet. The variable P is initially set to 0.5, so each of f and g is chosen approximately equally often.

Activity 5.6 Two-function systems (Optional)

Open Mathcad file **221B3-03 Iterated function systems**. Page 2 of the worksheet defines the matrices representing basic linear transformations, using the Mathcad notation.

Move to page 3. Here two affine transformations

$$f(\mathbf{x}) = \mathbf{Ax} + \mathbf{a} \quad \text{and} \quad g(\mathbf{x}) = \mathbf{Bx} + \mathbf{b},$$

are defined, where

$$\mathbf{A} = U(r1)\,R(\theta 1) \quad \text{and} \quad \mathbf{B} = U(r2)\,R(\theta 2).$$

The variables $r1$, $\theta 1$, $r2$ and $\theta 2$, and the vectors $\mathbf{a}$ and $\mathbf{b}$, are initially set to values that give the transformations f and g used to generate the sequence in Figure 5.3.

The initial point $\mathbf{x}_0$ is set to the origin. The iteration sequence $\mathbf{x}_n$ of points is generated by using a sequence p_n of random numbers and a variable P to decide which of the transformations f and g is used at each iteration, in the way described before this activity. The number of iterations carried out is N.

Each calculated point of the sequence is displayed on a graph as a small dot. The graph can be rescaled by changing the values of $s1$ and $s2$.

The number of iterations, N, is set to 5000. If your computer is fast enough, then you may wish to increase this to 20 000 or 50 000, for example, to obtain 'better' plots.

(a) Make the following three changes in turn (do not make other changes as you do this), and observe each new plot obtained.

(i) $r1 = 0.85$ (ii) $r1 = 0.7$ (iii) $\theta 1 = 0.3$

(b) Make the following changes, and observe the plot obtained after they are *all* made:

$$r1 = 0.95, \quad \theta 1 = 0.5, \quad r2 = 0.25, \quad P = 0.9.$$

(You may also wish to set $s1 = -1$.)

In part (b), Mathcad starts a new plot after each change. For ways to avoid waiting, see the Comment on page 46.

Comment

By varying the parameters $r1$, $\theta 1$, $r2$ and $\theta 2$, you can alter the shape of the sequence generated by the iterated function system and so produce a wide variety of 'natural-looking objects'.

Mathcad notes

See *A Guide to Mathcad*.

- ◇ The Mathcad function **if** chooses one of two values depending on a condition. The expression **if**(condition, a, b) gives the value a if the condition is true, and b if the condition is false. Several **if** statements can be combined together to make multiple choices.
- ◇ The Mathcad function **rnd** generates uniformly-distributed random numbers. The expression **rnd**(x) returns a random number between 0 and x. The values produced by the **rnd** function come from a sequence of random numbers generated by Mathcad when it starts up. The same sequence is generated every time, unless you change the random number generator. To do this, select **Worksheet Options...** from the **Tools** menu, choose the 'Built-In Variables' tab and then set the 'Seed value for random numbers'. Each choice of positive integer gives a different sequence; the default value is 1.
- ◇ The graphs in the worksheet are plotted using the trace type 'points', with the 'Symbol' column left blank. Mathcad plots each point as a small dot.

It is a remarkable fact that the shape of the set obtained by this method is associated only with f and g, and not with the way that we choose between them at each iteration. Briefly, if the functions f and g are both *contracting*, that is, they reduce the distance between pairs of points, then there is a bounded set F of points which has the property that if we choose any point in F, and apply *either* f or g to this point, then we obtain another point in F. If we choose a point outside F, and repeatedly apply either f or g, then the resulting sequence approaches the set F, and may or may not eventually lie within F.

One way to obtain a picture of the set F is to iterate f and g in the random manner described. This usually produces a 'chaotic' sequence whose points, at least after the first few, are either in the set or very close to the set. For some pairs of transformations f and g, such as the pair in Activity 5.6(b), we need to ensure that one of the transformations is applied more often than the other if we want the points of the sequence to be fairly evenly distributed across the set.

The self-similar nature of sets like those in Activity 5.6 led mathematicians to the remarkable discovery that it is possible to use iterated function systems to generate a *given* set by analysing the ways in which it is self-similar. This usually requires more than two affine transformations. A classic example is the fern, which you can see in the final activity.

Activity 5.7 The fern: a four-function system (Optional)

You should still be working with Mathcad file 221B3-03.

Move to page 4 of the worksheet, and observe the plot obtained from the given iterated function system of four affine transformations.

If your computer is fast enough, then you may wish to increase the number of iterations N to 20 000 or 50 000, for example, to obtain a better plot.

If you wish to experiment with the plot, then you could change the rotation matrix in the definition of the matrix $\mathbf{A}$; for example, try $R(0)$ and then $R(0.05)$. Other small changes to the parameters in the matrices will give ferns of slightly different shapes.

Now close Mathcad file 221B3-03.

Solutions to Activities

Chapter B1

Solution 4.2

(a) When c takes any of the values 0.5, 0.6 or 0.7, the sequence tends to infinity.

(b) When c takes either of the values -0.1 or 0.1, the sequence tends to a fixed point.

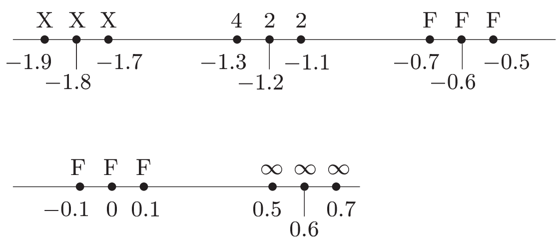
(c) When c takes any of the values -0.5, -0.6 or -0.7, the sequence tends to a fixed point.

(d) When c takes either of the values -1.1 or -1.2, the sequence tends to a 2-cycle.

When c takes the value -1.3, the sequence tends to a 4-cycle.

(e) When c takes any of the values -1.7, -1.8 or -1.9, the sequence seems to be chaotic.

The results of parts (a) to (e) are summarised in the following figure.



Solution 4.3

(a) The sequence x_n tends to a fixed point for all values of c in the interval $[-0.75, 0.25]$.

For values of c inside this interval but close to its left end, this convergence is difficult to see from the graphical iteration diagram or the table, but it is confirmed by the fact that the gradient of the relevant fixed point lies between -1 and 1, so the fixed point is attracting. Note that if c is inside the interval but very close to either end, then the gradient at the attracting fixed point is displayed as ± 1, due to rounding.

When $c = 0.25$ and $c = -0.75$ the relevant fixed points are indifferent, and in both cases you have to set Mathcad to carry out a large number of iterations to see that the sequence appears to tend to the fixed point.

For values of c a little greater than 0.25, the sequence tends to infinity.

For values of c a little less than -0.75, both fixed points are repelling, and the sequence tends to a 2-cycle.

(b) The sequence x_n tends to a 2-cycle for all values of c in the interval $[-1.25, -0.75)$.

When $c = -1.25$ the relevant 2-cycle is indifferent, and you have to set Mathcad to carry out a large number of iterations to see that the sequence appears to tend to the 2-cycle. For values of c a little less than -1.25, the 2-cycle is repelling, and the sequence tends to a 4-cycle.

Solution 4.4

(a) The behaviour of the sequence x_n for the given values of c is as follows:

-1.39: x_n tends to an 8-cycle;
-1.475: x_n tends to a 6-cycle;
-1.575: x_n tends to a 7-cycle;
-1.76: x_n tends to a 3-cycle.

(b) Setting $c = -1.4$ produces a graphical iteration diagram with a fairly large number of construction lines. Increasing the number of iterations does not produce any visible new construction lines, so it appears that the sequence tends to a cycle. The number of members of the cycle cannot be counted in the diagram scaled from -2 to 2, since some of the construction lines are too close together. (In fact, this is a 32-cycle.)

Solution 4.6

(b) Values of c between about -1.628 and -1.625 give 5-cycles.

Solution 4.7

(a) The three fixed points are 0, 0.766 and -0.766 (to three decimal places). They are all repelling.

(b) The two 2-cycles are -1.034, -0.467 and 1.034, 0.467 (to three decimal places). They are both attracting.

(c) When $x_0 = -1$, the sequence tends to the first 2-cycle given in the answer to part (b).

When $x_0 = 1$, the sequence tends to the second 2-cycle given in the answer to part (b).

Chapter B2

Solution 5.2

(c) (i) The matrix represents the composite of a scaling with factors 2 and 1, followed by an x-shear with factor 2; that is,

$$\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} = X(2)\,S(2,1) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

(You can check the answer by matrix multiplication.)

(ii) The matrix represents the composite of a scaling with factors 2 and 1, followed by an x-shear with factor 2, followed by a rotation through the angle π; that is,

$$\begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix} = R(\pi)\,X(2)\,S(2,1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

(iii) The matrix represents the composite of an x-shear with factor -1, followed by a rotation through the angle $\frac{1}{2}\pi$; that is,

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = R(\tfrac{\pi}{2})\,X(-1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

(iv) The matrix represents the composite of a scaling with factors -3 and 3, followed by an x-shear with factor $-\frac{1}{3}$, followed by a rotation through the angle $\frac{3}{2}\pi$ (or, equivalently, $-\frac{1}{2}\pi$); that is,

$$\begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix} = R(\tfrac{3\pi}{2})\,X(-\tfrac{1}{3})\,S(-3,3) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}.$$

Solution 5.3

Target figure 1 is obtained by setting

$$\mathbf{A} = R(\tfrac{\pi}{4}) \quad\text{and}\quad \mathbf{a} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Target figure 2 is obtained by setting

$$\mathbf{A} = Q(0) \quad\text{and}\quad \mathbf{a} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Target figure 3 is obtained by setting

$$\mathbf{A} = S(2,3) \quad\text{and}\quad \mathbf{a} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}.$$

Target figure 4 is obtained by setting

$$\mathbf{A} = Y(2) \quad\text{and}\quad \mathbf{a} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Chapter B3

Solution 5.1

(b) The ratio y_N/x_N appears to tend to 0 as N increases. This could have been predicted because 0 is the gradient of the dominant eigenline $y = 0$.

(c) If the initial point lies to the right of the y-axis, then the sequence tends in the positive direction of the x-axis, whereas if the initial point lies to the left of the y-axis, then the sequence tends in the negative direction of the x-axis.

(The dominant eigenvalue 1.5 is positive, so each point of the sequence lies on the same side of the non-dominant eigenline as the initial point.)

If the initial point lies on the y-axis, then subsequent points of the sequence coincide with the initial point. (In this case, Mathcad highlights the ratio y_N/x_N in red and displays no value for it, since its calculation would require division by zero.)

Solution 5.2

(a) The points of the sequence alternate between opposite sides of the eigenline $y = 4x$ and stay on the same side of the eigenline $y = -x$ as the initial point. The sequence moves away from $(0, 0)$ and tends in the direction of the dominant eigenline $y = -x$.

(b) The points of the sequence stay on the same side of the eigenline $y = x$ and on the same side of the eigenline $y = \frac{1}{3}x$ as the initial point. The sequence moves away from $(0, 0)$ and tends in the direction of the dominant eigenline $y = x$.

(c) The points of the sequence stay on the same side of the eigenline $y = -2x$ and on the same side of the eigenline $y = -3x$ as the initial point. The sequence moves towards $(0, 0)$ and tends in the direction of the dominant eigenline $y = -2x$.

(d) The points of the sequence stay on the same side of the eigenline $y = -\frac{3}{5}x$ as the initial point, and alternate between opposite sides of the other eigenline $y = 3x$. The sequence moves towards $(0, 0)$. Neither eigenline is dominant.

(In fact, the points lie alternately on each of a pair of lines through the origin.)

Solution 5.3

(a) The matrix $R(\frac{\pi}{6})$ gives a sequence each of whose points is one of twelve points equally spaced around a circle centred at the origin.

The matrix $R(\frac{\pi}{3})$ gives a sequence each of whose points is one of six points equally spaced around a circle centred at the origin.

The matrix $R(1)$ gives a sequence whose points lie around a circle centred at the origin; no points are repeated.

(b) The matrix $U(0.9)\,R(\frac{\pi}{12})$ gives a sequence whose points lie on a spiral centred at the origin; each point is closer to the origin than its predecessor.

The matrix $U(-0.9)\,R(\frac{\pi}{12})$ gives a sequence whose points lie alternately on each of two spirals centred at the origin; each point is closer to the origin than its predecessor.

The matrix $U(1.01)\,R(\frac{\pi}{12})$ gives a sequence whose points lie on a spiral centred at the origin; each point is further from the origin than its predecessor.

(c) The matrix $S(0.8, 1.2)\,R(\frac{\pi}{12})$ gives a sequence whose points lie on an 'elliptical spiral', centred at the origin and spiralling towards the origin.

(d) The matrix $X(0.2)\,R(\frac{\pi}{12})$ gives a sequence whose points lie on an ellipse centred at the origin.

The matrix $X(-0.6)\,R(\frac{\pi}{12})$ also gives a sequence whose points lie on an ellipse centred at the origin.

Solution 5.4

The value of the ratio y_N/x_N seems to approach the gradient of the eigenline in all six cases.

(It is indeed true that if $\mathbf{A}$ is a 2×2 matrix with just one eigenvalue and just one eigenline, and $\mathbf{x}_0$ is a point in the plane, then

$$\frac{y_n}{x_n} \to m \text{ as } n \to \infty,$$

where (x_n, y_n) is the $(n+1)$th point of the iteration sequence $\mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0$ and m is the gradient of the eigenline.)

Solution 5.5

(a) Setting $N = 20$ illustrates that after the first twelve points the sequence begins to repeat. Each point in the sequence is one of twelve points equally spaced on a circle centred at the origin. (This sequence was the first of those considered in Activity 5.3(a).)

Changing $\mathbf{a}$ to a vector other than the position vector of the origin produces a sequence whose points lie on a circle centred at a point other than the origin.

(Changing $\mathbf{a}$ usually also changes the radius of the circle, since this is the distance between the centre of the circle and the initial point.)

(b) Setting $N = 40$ plots more points of the sequence, spiralling towards the origin.

Changing $\mathbf{a}$ to a vector other than the position vector of the origin produces a sequence that spirals inwards about a point other than the origin.

Computer Book C
Calculus

Guidance notes

This computer book contains those sections of the chapters in Block C which require you to use Mathcad. Each of these chapters contains instructions as to when you should first refer to particular material in this computer book, so you are advised not to work on the activities here until you have reached the appropriate points in the chapters.

In order to use this computer book, you will need the following Mathcad files.

Chapter C1

221C1-01 Differentiation
221C1-02 A differentiation template
221C1-03 The Newton-Raphson method

Chapter C2

221C2-01 Integration templates
221C2-02 Volumes of solids of revolution (Optional)

Chapter C3

221C3-01 Taylor series and polynomials

Instructions for installing these files onto your computer's hard disk, and for opening them, are given in Chapter A0 of MST121.

The computer activities for Chapter C2 also require you to work with Mathcad worksheets which you have created yourself.

Activities based on software vary both in nature and in length. Sometimes the instructions for an activity appear only in the computer book; in other cases, instructions are given in the computer book and on screen. Feedback on an activity is sometimes provided on screen and sometimes given in the computer book.

For advice on how each computer session fits into suggested study patterns, refer to the Study guides in the chapters.

Chapter C1, Section 5 Differentiation with the computer

Mathcad can be used to find the derivatives of functions, which is the topic of Subsection 5.1. In Subsection 5.2, you will see the Newton–Raphson method implemented on the computer.

5.1 Finding derivatives

In Mathcad, derivatives are found using the $\frac{d}{dx}$ operator. This operator can be used both to find a general formula for the derivative of a function f, and to find the numerical value of the derivative of f at a particular point. Both these usages are introduced in this subsection.

Activities 5.1 and 5.2 below repeat activities in MST121 in which the Mathcad approach to finding derivatives is introduced, and their associated Mathcad file is identical to the corresponding file used in MST121. If you have recently studied Chapter C1 of MST121, then you should not find it necessary to work through Activities 5.1 and 5.2 in detail, though you may wish to refresh your memory. This also applies to the comments on each activity. If you have not recently studied Block C of MST121, then you should work through Activities 5.1 and 5.2 in the normal way.

Activity 5.1 Finding a formula for the derivative

Remember to create your own working copy of the file.

Open Mathcad file **221C1-01 Differentiation**. Page 1 introduces the worksheet. Work through page 2, and then carry out Task 1 on page 3.

Solutions are given on page 96.

Comment

- ◇ Make sure that the variables entered in the two placeholders of the $\frac{d}{dx}$ operator match. For example, if you mistakenly try to evaluate symbolically the expression

$$\frac{d}{dt}\cos(4x),$$

then you will obtain the answer 0.
- ◇ Brackets are necessary when entering an expression of more than one term into the right-hand placeholder of the $\frac{d}{dx}$ operator. However, if the expression $x^3 - 6x^2 - 15x + 54$ is entered into the right-hand placeholder, then Mathcad will automatically add appropriate brackets when the first plus or minus sign is typed.
- ◇ Sometimes the expression for a derivative obtained by Mathcad can be 'improved' by simplifying it. In place of symbolic evaluation ($\rightarrow$), either of the symbolic keywords 'simplify' and 'factor' can be applied to obtain a derivative. The outcome from these may or may not be the same as that from using $\rightarrow$, but will sometimes be in a more convenient form. However, what Mathcad regards as 'simpler' may not necessarily seem so to a human observer!

These symbolic keywords were introduced in MST121 Chapter A0, file 121A0-05. See also *A Guide to Mathcad* for further details.

Mathcad notes

◇ The expressions e^x and $\exp(x)$ are equivalent in Mathcad, but the former is always used in the output of symbolic calculations, irrespective of the form used for input. Similarly, the square root sign appears in output rather than the power $\frac{1}{2}$ (although $x^{3/2}$ is used rather than $x\sqrt{x}$, etc.).

◇ If Mathcad is unable to carry out a symbolic operation (there are many possible reasons for this), then it will reproduce the input expression unchanged, apart from possible notational changes of the type just noted.

Remember that Mathcad notes are *optional*.

Activity 5.1 showed how to use the $\frac{d}{dx}$ operator and symbolic evaluation to obtain an algebraic expression for the derivative. This replicates what you might do by hand, but Mathcad can also be used to differentiate functions that would be rather complicated to do by hand.

We turn next to how Mathcad can be applied to find the numerical value for the derivative at a particular point.

Activity 5.2 Evaluating the derivative at a point

Turn to page 4 of the worksheet, and carry out Task 2.

Solutions are given further down page 4 of the worksheet.

You should still be working with Mathcad file 221C1-01.

Comment

◇ Note the comments made towards the bottom of page 4 of the worksheet. Once a value has been defined for the differentiation variable (x, say), then the $\frac{d}{dx}$ operator can be evaluated either symbolically ($\frac{d}{dx}(\ldots) \rightarrow$) or numerically ($\frac{d}{dx}(\ldots) =$) to find the numerical value of the derivative at that particular value of x.

  These two evaluation methods usually give the same numerical value, but Mathcad calculates the two results in different ways. For symbolic evaluation, Mathcad finds a general formula for the derivative and then evaluates this formula for the particular value of x, whereas for numerical evaluation, Mathcad uses a numerical algorithm to find an approximate decimal value.

Essentially, this involves a direct application of the definition of derivative, as given by equation (1.2) in Chapter C1. See also the second Mathcad note overleaf.

◇ If it is desired to ‘turn off’ a numerical value assigned previously to x (say), so as to obtain an algebraic expression from symbolic evaluation of a derivative with respect to x, this can be achieved by inserting the assignment $x := x$ just before the derivative.

◇ If the expression $\frac{d}{dx}f(x)$ is entered, where a definition for the function $f(x)$ is provided earlier in the worksheet, then Mathcad can find either a symbolic derivative for $f(x)$ or the numerical value of this derivative at any specified point, just as before.

Mathcad notes

- ◇ When you enter $\rightarrow$ to evaluate an expression symbolically, or $=$ to evaluate it numerically, it doesn't matter where on the expression the blue editing lines are. All that matters is that the expression is complete, with every placeholder filled in. These two evaluation methods also behave in a similar way if a change is made to the worksheet, above or to the left of a calculation. In automatic calculation mode (the default), the result of a calculation involving either $\rightarrow$ or $=$ is updated automatically, while in manual mode, you can press the `[F9]` key to update the result.
- ◇ When evaluating a derivative numerically ($\frac{d}{dx}(\ldots) =$), you must define earlier in the worksheet the point at which the derivative is to be found; for example, $x := 3$. Mathcad then uses a numerical algorithm to obtain an approximation to the exact value of the derivative at that point, which is usually accurate to 7 or 8 significant figures. Very occasionally the method fails, in which case the derivative is highlighted in red. Clicking on this expression reveals the error message 'This calculation does not converge to a solution.'.

To change from automatic to manual calculation mode, or vice versa, first choose **Calculate** from the **Tools** menu, then click on **Automatic Calculation**.

Without this prior definition, x appears in red in the $\frac{d}{dx}$ operator, as an undefined variable.

Now close Mathcad file 221C1-01.

Mathcad provides a useful means of checking differentiations that you have carried out by hand. However, an answer supplied by Mathcad can look quite different from an answer obtained by hand, even if you use Mathcad to simplify the expression. The next activity gives you practice in checking that answers supplied by Mathcad are equivalent to answers obtained by hand.

The Mathcad file associated with this activity provides a 'differentiation template' that you may also wish to use in other circumstances.

Activity 5.3 Further derivatives

Open Mathcad file **221C1-02 A differentiation template**. The worksheet consists of a single page, which can be used to differentiate any function $f(x)$ that you input. The page is set up with the function

$$f(x) = \frac{x^2+1}{2x-1}.$$

Note that the output from applying the symbolic keywords 'simplify' and 'factor' to the derivative of this function is different from that obtained by straightforward symbolic evaluation.

Note also, towards the bottom of the page, that the template can also be used to provide a numerical value for the derivative $f'(x)$ at any specified point at which the derivative exists.

While the facilities for obtaining numerical values are included in the template for completeness, they do not otherwise feature in this activity.

Each of parts (a)–(e) on the next page gives a function $f(x)$ that you were asked to differentiate in the main text, together with the answer for $f'(x)$ provided in the solutions. In each case, enter the formula for the function into the definition of $f(x)$ in the template, and note the results. If it is not clear that one of the answers supplied by Mathcad is equivalent to the stated expression for $f'(x)$, then copy the closest-looking Mathcad answer onto a piece of paper and carry out further algebraic manipulation by hand to confirm the equivalence.

Remember to enter each multiplication, and to enclose the arguments of trigonometric functions in brackets. For example, the expressions for $f(x)$ in parts (a) and (c) can be entered by typing `x*e∧x∧2` and `cos((x+4)*sec(x))` , respectively. The square root symbol required for part (b) can be obtained from a button on the 'Calculator' toolbar, or by typing \ (backslash).

(a) $f(x) = xe^{x^2}, \qquad f'(x) = (2x^2 + 1)e^{x^2}$ — See Activity 2.6(a).

(b) $f(x) = \dfrac{\sin(\sqrt{x})}{\sqrt{x}}, \qquad f'(x) = \dfrac{\sqrt{x}\cos(\sqrt{x}) - \sin(\sqrt{x})}{2x^{3/2}}$ — See Activity 2.6(b).

(c) $f(x) = \cos((x+4)\sec x)$, — See Activity 2.6(c).

$f'(x) = -\sec x(1 + (x+4)\tan x)\sin((x+4)\sec x)$

(d) $f(x) = x^4 e^x \sin x, \qquad f'(x) = x^3 e^x((4+x)\sin x + x\cos x)$ — See Activity 2.4(a).

(e) $f(x) = \dfrac{x\tan x}{1+x^2}, \qquad f'(x) = \dfrac{x(1+x^2)\sec^2 x + (1-x^2)\tan x}{(1+x^2)^2}$ — See Activity 2.4(b).

Solutions are given on page 96.

Mathcad notes

It is safest to enter all products explicitly in Mathcad, by clicking on the 'Multiplication' × button on the 'Calculator' toolbar or by typing `*`. In some situations, for example when using brackets, Mathcad will assume that you intended to enter a product, and will insert the multiplication for you. However, Mathcad will *not* help in this way if you enter `xe` rather than `x*e` in part (a). In this case, Mathcad assumes that you have entered a single variable name, 'xe', rather than the product of the variable x and the exponential constant e.

Now close Mathcad file 221C1-02.

Sometimes you may want to use Mathcad to find a second-order derivative. One way to do this is by using the $\frac{d}{dx}$ operator twice. You can enter the $\frac{d}{dx}$ operator in your worksheet, then enter the $\frac{d}{dx}$ operator again into the right-hand placeholder, and then fill in all the placeholders appropriately. For example, Mathcad gives the result

$$\frac{d}{dx}\left(\frac{d}{dx}(x^3 - 6x^2 - 15x + 54)\right) \;\rightarrow\; 6x - 12.$$

Another way to find a second-order derivative in Mathcad is to use the $\frac{d^n}{dx^n}$ operator, for which there is a button on the 'Calculus' toolbar. You should enter '2' in the bottom index placeholder (which will cause a 2 to appear in the top placeholder as well), and fill in the other placeholders just as for the $\frac{d}{dx}$ operator. You can evaluate the resulting expression either symbolically or numerically, in the same way as for expressions involving the $\frac{d}{dx}$ operator. For example, Mathcad gives the result

The keyboard alternative is `[Ctrl]?` (given by the three keys `[Ctrl]`, `[Shift]` and `/`).

$$\frac{d^2}{dx^2}(x^3 - 6x^2 - 15x + 54) \;\rightarrow\; 6x - 12.$$

5.2 The Newton–Raphson method

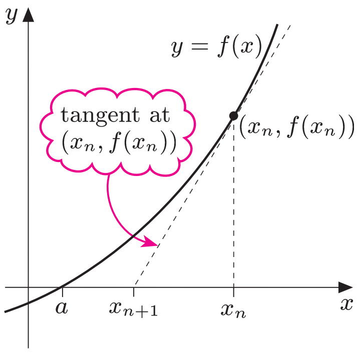


Figure 5.1

You saw in the main text that a solution of the equation $f(x) = 0$ is called a 'zero' of the function f, and that it is often possible to find an approximation to a zero of a smooth function f by using the Newton–Raphson method. The method is to start with an initial term x_0 and calculate the iteration sequence given by the recurrence relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n = 0, 1, 2 \ldots).$$

Each term x_{n+1} in this sequence is the x-coordinate of the point where the tangent at $(x_n, f(x_n))$ to the graph of f cuts the x-axis, as illustrated in Figure 5.1. Usually each term of the sequence is a better approximation to a solution of $f(x) = 0$ than the preceding term.

The next activity shows how the Newton–Raphson method can be applied in Mathcad to an equation that you saw in Activities 4.1 and 4.3 of the main text.

Activity 5.4 The Newton–Raphson method on the computer

Open Mathcad file **221C1-03 The Newton-Raphson method**. The worksheet consists of a single page, in which the terms of a Newton–Raphson iteration sequence are calculated and displayed on a graph and in a table (of the last eleven terms).

The page is set up with the function f defined by $f(x) = x^3 - 2x - 2$. The number of iterations, N, is set to 0, and the initial term, x_0, is set to 2.

The notation Df is used rather than f' because the prime symbol is difficult to see in Mathcad when placed immediately to the right of an 'f'. (More information about the definition of $Df(x)$ is given in the Mathcad note on the next page.)

The worksheet uses the name Df rather than f' for the derived function of f in the Newton–Raphson recurrence relation. The definition of $Df(x)$ is near the top of the page; it uses the $\frac{d}{dx}$ operator to obtain a formula for the derivative of $f(x)$.

(a) (i) Set $N = 1$. The first iteration of the Newton–Raphson method is displayed on the graph. The first two terms x_0 and x_1 of the resulting sequence are displayed in a table, together with the values of f at these points.

(ii) Set $N = 2$ to see the next iteration.

(iii) Set $N = 5$. The resulting further iterations cannot be seen on the graph because the construction lines are very close to the point where the curve crosses the x-axis. However, the values of the terms calculated are displayed in the table.

Use these values to state an approximation for the zero of f which can be seen on the graph.

(b) (i) Change the graph x-axis limits to $X1 = -1.6$ and $X2 = 2.6$, to display more of the graph of f. Then set $x_0 = 0.82$. Describe what you observe.

(ii) Set $N = 15$. The table now shows the terms x_5 to x_{15}, and you should notice new blue construction lines within the graph box, closing in on the zero of f.

Find the smallest value of n for which $f(x_n) = 0$ to nine decimal places.

You considered the initial terms $x_0 = 0.8165$ and $x_0 = 0$ in Activity 4.3 of the main text. You may also try $x_0 = 0.8165$ here, but Mathcad has difficulty in drawing all of the blue construction lines that should appear within the graph box. The smallest value of n with $f(x_n) = 0$ to 9 d.p. in this case is $n = 34$.

(c) Set $x_0 = 0$ and describe what you observe.

Solutions are given on page 96.

Mathcad notes

In the worksheet, the function Df is defined as the result of a symbolic calculation; for example,

$$Df(x) := \frac{d}{dx}f(x) \;\rightarrow\; 3x^2 - 2.$$

This expression may look a little strange, but it is just a combination of some basic Mathcad features and operators. It consists of a standard function definition for $Df(x)$, the $\frac{d}{dx}$ operator to calculate the derivative of f, symbolic evaluation $\rightarrow$ and the symbolic result. It is this result which becomes the right-hand side of the definition for $Df(x)$. If any change is made to f, then Df will change also.

In Activity 5.4(b) and (c), you saw examples of difficulties with the Newton–Raphson method. You should be reassured, however, that such cases are the exception rather than the rule. For most functions and initial terms, the Newton–Raphson method produces a sequence that rapidly approaches a zero of the function. In particular, the method is very reliable if you choose an initial term x_0 that is fairly close to the zero which you seek.

You can choose a suitable value for x_0 by using a sketch or Mathcad plot of the graph of the function.

Activity 5.5 Using the Newton–Raphson method

You should still be working with Mathcad file 221C1-03.

For each of the functions f in parts (a)–(c) below, use the Mathcad worksheet to find approximations for all the solutions of the equation $f(x) = 0$ in the given interval.

In each part, start by resetting the number of iterations N to 0, then edit the definition of the function f to set it to the given function, and set the graph x-axis limits to the endpoints of the given interval. Then choose an appropriate initial term x_0, and set N large enough to display a term x_n for which $f(x_n) = 0$ to nine decimal places.

Where there is more than one zero in the given interval, you will need to choose an appropriate initial term x_0 to find each one.

(a) $f(x) = x^2 - x - 1$, in the interval $[1, 2]$

(b) $f(x) = x^3 - x^2 - 5x + 3$, in the interval $[-3, 3]$

(c) $f(x) = x + e^x$, in the interval $[-1, 0]$

Solutions are given on page 96.

Comment

It is possible for a Newton–Raphson iteration sequence with initial term x_0 to converge to a zero other than the zero closest to x_0. For example, in part (b) the initial value $x_0 = 1.5$ lies between the two positive zeros of f but the sequence x_n converges to the negative zero.

Mathcad notes

The recurrence relation in this worksheet incorporates a check on whether the derivative $Df(x_n)$ is close to zero, to avoid division by zero or by a very small number. If $|Df(x_n)| \leq 10^{-9}$ for any value of n, then the current value of x_n is repeated subsequently, and there will not be convergence to a zero of f. In such a case, choose a different value for x_0.

Now close Mathcad file 221C1-03.

Chapter C2, Section 5
Integration with the computer

Mathcad can be used to perform integration. In Subsections 5.1 and 5.2 you will use Mathcad to find indefinite and definite integrals, respectively. There is one prepared Mathcad file for these subsections, which provides 'integration templates', but you will also create your own worksheets.

Subsection 5.3 is *optional*; it involves using a prepared Mathcad file to explore estimates for the volumes of solids of revolution obtained by summing the volumes of cylinders.

5.1 Finding indefinite integrals

In Mathcad, indefinite integrals are found using the $\int$ operator. Like the $\frac{d}{dx}$ operator, the $\int$ operator can be used with Mathcad's symbolic commands, to find an algebraic expression for an integral of a given function.

In this subsection you are invited to find indefinite integrals for a variety of functions. Activity 5.1 repeats an activity in MST121 in which the Mathcad approach to finding indefinite integrals is introduced. If you have recently studied Chapter C2 of MST121, then you should not find it necessary to work through Activity 5.1 in detail, though you may wish to refresh your memory. This also applies to the comments on the activity.

Activity 5.1 How to find indefinite integrals

In this activity you will use Mathcad to find the indefinite integral of x^2.

The buttons referred to below are on the 'Calculus' and 'Symbolic' toolbars. If you wish to use these and they are not already visible, then either click on the appropriate buttons on the 'Math' toolbar, or select the **View** menu, **Toolbars** and choose **Calculus**, then repeat and choose **Symbolic**.

If necessary, see Activity 6.3(a) for Chapter A2 in Computer Book A, or consult *A Guide to Mathcad*.

Be careful not to confuse the $\int$ button with the $\int_a^b$ button, which is used for finding definite integrals (as you will see later).

(a) Create a new (Normal) worksheet.

(b) Enter the $\int$ operator, either by clicking on the $\int$ button on the 'Calculus' toolbar, or by using the keyboard alternative `[Ctrl]i`.

(c) Enter the expression to be integrated, x^2, in the first placeholder after the integral sign. (This expression is called the integrand.) Then enter the variable of integration, x, in the placeholder after the 'd'.

(d) Click on the $\rightarrow$ button ('Symbolic Evaluation') on the 'Symbolic' toolbar, or use `[Ctrl].`, the keyboard alternative. Then click elsewhere on the page, or press `[Enter]`, to obtain the integral. Check that the answer provided by Mathcad is what you expect.

(e) Now go through the same procedure to evaluate the integral $\displaystyle\int u^2\,du$.

(f) If you wish to save your work, then select the **File** menu and use **Save As...** to name and save your worksheet. (It is a good idea to insert a title in your worksheet. If you need to create space for this, then do so by positioning the red cross cursor at the top of the worksheet and pressing `[Enter]` to insert as many blank lines as required.)

Comment

- Notice that the $\int$ operator in Mathcad gives only *an* integral (that is, an antiderivative) of the integrand. It does not give *the indefinite* integral because it does not add an arbitrary constant.
- The outcomes from integrating $\int x^2\,dx$ and $\int u^2\,du$ demonstrate that the form of the indefinite integral does not depend on the choice of symbol for the variable of integration.

Mathcad notes

- In part (e), it is sufficient simply to edit the integral expression $\int x^2\,dx$, replacing each 'x' by a 'u'.
- You can also find an integral of a function f by first defining f and then evaluating symbolically the expression $\int f(x)\,dx$.
- The behaviour of the $\int$ operator differs somewhat from that of the $\frac{d}{dx}$ operator, used in Chapter C1. The $\int$ operator *cannot* be evaluated numerically (by typing `=`), since this makes no sense in the context of finding an indefinite integral. If you try this, after setting a value for the variable of integration, then the integral is highlighted in red. Clicking on the integral reveals the error message 'This operator must be evaluated symbolically.'.

  However, once x has been assigned a value, symbolic evaluation of an indefinite integral with respect to x does give a numerical answer in Mathcad, just as for a derivative. This unwanted behaviour can be prevented by typing $x := x$ just before the indefinite integral.

A similar usage of $x := x$ was referred to in the context of derivatives, on page 73.

In the next activity, you should either continue with the worksheet created in Activity 5.1 or create a new (Normal) worksheet.

Activity 5.2 Finding indefinite integrals

Each of parts (a)–(c) below gives an indefinite integral that you were asked to find in the main text, together with the answer provided in the solutions.

In each part, use Mathcad to find an integral, and compare the answer given by Mathcad with the answer stated here.

If necessary, refer to Activity 5.1(b)–(d).

(a) $\displaystyle\int (u + 2e^{7u})\,du = \tfrac{1}{2}u^2 + \tfrac{2}{7}e^{7u} + c$

See Activity 1.3(c).

(b) $\displaystyle\int \cos^2 x\,dx = \tfrac{1}{4}\sin(2x) + \tfrac{1}{2}x + c$

See Activity 1.3(e).

(c) $\displaystyle\int \frac{x^2}{1+x^3}\,dx = \tfrac{1}{3}\ln|1+x^3| + c$

See Activity 3.3(e).

In part (b), $\cos^2 x$ should be input as $\cos(x)^2$; for example, type `cos(x)^2`.

Solutions are given on page 97.

Comment

- After the '$+\,c$' has been added, the Mathcad answers to parts (a) and (b) agree with those obtained in the main text, and that for part (c) is equivalent when $1 + x^3 > 0$.
- Recall (from the context of obtaining derivatives symbolically) that the symbolic keywords 'simplify' and 'factor' can be used as alternatives to $\rightarrow$. When applied to integrals, either of these may again provide an outcome in a more convenient form.

See the final Comment item for Chapter C1, Activity 5.1 on page 72 of this computer book.

Save the worksheet that you have created, if you wish. Then close the file.

Mathcad provides a useful means of checking integrations that you have carried out by hand. However, an answer supplied by Mathcad can look quite different from an answer obtained by hand, even if you use Mathcad to simplify the expression. The next activity gives you practice in checking that answers supplied by Mathcad are equivalent to answers obtained by hand.

The Mathcad file associated with this activity provides a 'template for indefinite integrals' that you may also wish to use in other circumstances.

Activity 5.3 Further indefinite integrals

Open Mathcad file **221C2-01 Integration templates**, and turn to page 2 of the worksheet. This page can be used to find an integral (antiderivative) of any function $f(x)$ that you enter.

Note that the output from applying the symbolic keywords 'simplify' and 'factor' to the integral of a function is provided, as well as that obtained by straightforward symbolic evaluation.

Each of parts (a)–(c) below gives an indefinite integral that you were asked to find in the main text, together with the answer provided in the solutions. In each case, enter the formula for the integrand into the definition of $f(x)$ in the template, and note the results. If it is not clear that one of the answers supplied by Mathcad is equivalent to the stated expression for $\int f(x)\,dx$ (apart from omission of the '$+\,c$' in Mathcad), then copy the closest-looking Mathcad answer onto a piece of paper and try to carry out further algebraic manipulation by hand to confirm the equivalence. (However, do not spend long on this for part (c).)

See Activity 2.7(a).

(a) $\displaystyle\int x^3 \ln x\,dx = \tfrac{1}{16}x^4(4\ln x - 1) + c$

See Activity 3.5(c).

(b) $\displaystyle\int \frac{2x+3}{(x+2)^{1/3}}\,dx = \tfrac{6}{5}(x+2)^{5/3} - \tfrac{3}{2}(x+2)^{2/3} + c$

See Activity 3.6.

(c) $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx = \tfrac{1}{2}\arctan x + \tfrac{1}{4}\sin(2\arctan x) + c$

Comment

(a) The Mathcad answer resulting from 'factor' is closest to the stated answer, though the others are also clearly equivalent.

(b) The Mathcad answer from just symbolic evaluation can be seen to be equivalent to the stated answer. It is less obvious that this is the case for the Mathcad output from 'simplify' and 'factor', but algebraic manipulation, after taking $\frac{1}{10}(x+2)^{2/3}$ as a common factor in both terms of the expression given above, confirms that this is so. The 'simplify' output looks the simplest.

(c) Symbolic evaluation in Mathcad provides the answer

$$\int \frac{1}{(1+x^2)^2}\,dx \;\rightarrow\; \frac{\mathrm{atan}(x)}{2} + \frac{x}{2x^2+2},$$

and the output from 'simplify' and 'factor' is seen to be equivalent to this. Comparison with the given answer shows that the Mathcad output includes the expression

$$\frac{x}{2x^2+2} \quad \text{instead of} \quad \tfrac{1}{4}\sin(2\arctan x).$$

Note that Mathcad uses the notation 'atan' for arctan.

It is possible that your output will display terms in a different order to that shown here.

It is not immediately clear that these two expressions are equivalent, but in fact they are, as the following argument shows. For any $x \in \mathbb{R}$, let $\theta = \arctan x$. Then $x = \tan\theta$, so

$$\begin{aligned}\frac{x}{2x^2+2} &= \frac{1}{2}\left(\frac{\tan\theta}{1+\tan^2\theta}\right)\\ &= \frac{1}{2}\left(\frac{\sin\theta/\cos\theta}{\sec^2\theta}\right)\\ &= \tfrac{1}{2}\sin\theta\cos\theta\\ &= \tfrac{1}{4}\sin(2\theta) = \tfrac{1}{4}\sin(2\arctan x).\end{aligned}$$

This argument uses the trigonometric identities

$$1+\tan^2\theta = \sec^2\theta$$

and

$$\sin(2\theta) = 2\sin\theta\cos\theta.$$

This example illustrates that it may be difficult to tell whether an answer supplied by Mathcad is equivalent to one found by hand.

You can obtain evidence that two expressions *may* be equivalent by evaluating them at a few values of x, or by plotting both of them on the same Mathcad graph.

The next activity demonstrates that Mathcad's facility for finding indefinite integrals is not always as helpful as you might hope.

Activity 5.4 Mathcad does not always give the 'best' answer!

You should still be working with Mathcad file 221C2-01, on page 2 of the worksheet.

(a) In the main text, the indefinite integral

$$\int x\ln x\,dx = \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 + c$$

was obtained. Find this indefinite integral using Mathcad, and comment on the result.

See Chapter C2, Activity 2.2(a).

(b) The result

$$\int \sec^3 x\tan x\,dx = \tfrac{1}{3}\sec^3 x + c \quad (-\tfrac{1}{2}\pi < x < \tfrac{1}{2}\pi)$$

can be found by hand, using the substitution $u = \sec x$. Find this indefinite integral using Mathcad, and comment on the result.

(c) Use Mathcad to find the indefinite integral

$$\int x^n\,dx.$$

Comment on any problems with Mathcad's answer that you notice.

Comment

(a) Mathcad (with any of the three forms of output) gives the result $x^2\ln(x)/2$. This differs from the given expression by the non-constant term $-\frac{1}{4}x^2$, showing that here the Mathcad output is plain wrong! (This is despite the fact that Mathcad provides a correct general formula for $\int x^n\ln x\,dx$ for $n \neq -1$.)

(b) Mathcad gives an expression that is far more complicated than the one found by hand. It can be seen that each of the expressions provided by Mathcad is equivalent to

$$-\frac{\left(\tan^2(\frac{1}{2}x)+1\right)^3}{3\left(\tan^2(\frac{1}{2}x)-1\right)^3},$$

which is equal to the given expression $\frac{1}{3}\sec^3 x$ provided that

$$\sec x = \frac{1+\tan^2(\frac{1}{2}x)}{1-\tan^2(\frac{1}{2}x)}. \tag{5.1}$$

See Chapter A3, Subsection 3.3.

To establish this result, we use a double-angle formula, to obtain

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta\,(1-\tan^2\theta)\,.\end{aligned}$$

It follows that

$$\begin{aligned}\sec(2\theta) &= \frac{1}{\cos(2\theta)} \\ &= \frac{1}{\cos^2\theta\,(1-\tan^2\theta)} \\ &= \frac{\sec^2\theta}{1-\tan^2\theta} \\ &= \frac{1+\tan^2\theta}{1-\tan^2\theta}.\end{aligned}$$

The trigonometric identity

$$1+\tan^2\theta = \sec^2\theta$$

is applied here.

Putting $\theta = \frac{1}{2}x$ now gives equation (5.1).

The Mathcad answer is therefore equivalent to that found by hand, but it is unhelpful since the result appears in such a complicated form.

In the function definition $f(x) := x^n$ the variable n will appear in red, as it has not been defined. However, Mathcad ignores this error and is still able to evaluate $\int f(x)\,dx$ symbolically.

(c) Mathcad gives an answer that is correct for all values of n except $n = -1$, for which the answer provided involves a division by zero.

This highlights the fact that, when asked to obtain general results, Mathcad does so without regard to possible exceptions to the rule.

Now close Mathcad file 221C2-01.

5.2 Finding definite integrals

In Mathcad, definite integrals are found using the $\int_a^b$ operator. This operator can be used in either of two ways.

◇ *Symbolically*: Mathcad finds an algebraic expression for an antiderivative of the integrand, evaluates this at the upper and lower limits of integration, and subtracts the second value from the first.

◇ *Numerically*: Mathcad uses a numerical method to find an approximate value for the definite integral.

Examples of both techniques are given in this subsection.

In Activities 5.5–5.7 below, as in the first part of Subsection 5.1, you are asked to work in a new worksheet that you have created yourself.

You will need to decide whether you wish to save this worksheet.

Activity 5.5 introduces you to the symbolic use of the $\int_a^b$ operator. If you have recently studied Block C of MST121, then you should not need to do this activity, as it repeats an activity there, though you may wish to refresh your memory.

Activity 5.5 How to evaluate definite integrals

In this activity you will use Mathcad to evaluate the definite integral

$$\int_2^3 \frac{1}{x}\,dx.$$

(a) Enter the $\int_a^b$ operator in your worksheet, either by clicking on the $\int_a^b$ button on the 'Calculus' toolbar, or by using the keyboard alternative `&` (the ampersand sign, given on the keyboard by `[Shift]7`).

(b) Enter the integrand, the variable of integration, and the lower and upper limits of integration in the appropriate placeholders.

The `[Tab]` key provides a good way of moving around the placeholders.

(c) Click on the → button ('Symbolic Evaluation') on the 'Symbolic' toolbar, or use `[Ctrl].`, the keyboard alternative. Then click elsewhere on the page, or press `[Enter]`. You should obtain the answer

$$\ln(3) - \ln(2).$$

(d) Click anywhere on this answer, and then enter `=`, either by clicking on the `=` button on the 'Calculator' toolbar or by typing `=`, to evaluate it numerically. You should obtain the answer 0.405, which is the value of $\ln 3 - \ln 2$ to three decimal places.

Comment

Evaluating symbolically an expression involving the $\int_a^b$ operator gives an expression which is an *exact* answer (unless the original expression contains a decimal point; see the second Mathcad note below). In the example in this activity the expression is $\ln(3) - \ln(2)$. You can display the decimal value of such an expression by evaluating it numerically. This is done by selecting the expression, and then entering `=`.

Mathcad notes

◇ When you evaluate numerically an expression in Mathcad, the number of decimal places displayed is determined by the value of 'Number of decimal places'. The default value of this is 3, but you can change it by choosing **Result...** from the **Format** menu and then the 'Number Format' tab.

◇ If a Mathcad expression involving the $\int_a^b$ operator has a decimal point in any constant in the integrand, or in either limit of integration, then evaluating the expression symbolically gives a decimal answer with up to 20 decimal places. (Such an answer is unaffected by the value of 'Number of decimal places'.) For example, evaluating symbolically the integral below in Mathcad gives the outcome

$$\int_2^3 \frac{1.0}{x}\,dx \;\rightarrow\; 0.40546510810816438198.$$

In the next activity you will use the $\int_a^b$ operator to check the answers to some definite integrals that you were asked to evaluate by hand in the main text.

Activity 5.6 Evaluating definite integrals

If necessary, refer to the instructions in Activity 5.5.

Each of parts (a)–(d) below gives a definite integral that you were asked to evaluate in the main text. In each case, use symbolic evaluation in Mathcad to find an exact answer, and then evaluate this answer numerically, to obtain a value to three decimal places.

(Remember that π can be obtained from a button on the 'Calculator' or 'Greek' toolbar, or by typing `[Ctrl][Shift]p`.)

See Activity 1.5(a).

(a) $\displaystyle\int_0^1 \sin(\pi x)\,dx$

See Activity 1.5(c).

(b) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2 t\,dt$

See Activity 2.4(c).

(c) $\displaystyle\int_1^2 xe^{2x}\,dx$

See Activity 2.6(b).

(d) $\displaystyle\int_{\pi/12}^{\pi/6} x\sin(3x)\,dx$

Solutions are given on page 97.

In the next activity you will see an example of a definite integral that Mathcad is unable to evaluate symbolically. When this happens it is often worth attempting to evaluate the integral numerically, and the activity shows you how to do this. If you have recently studied Block C of MST121, then you should not need to do Activity 5.7, as it repeats an activity there, though you may wish to refresh your memory.

Activity 5.7 An awkward definite integral

(a) Use Mathcad to try to evaluate symbolically the definite integral

$$\int_0^1 e^{-t^3}\,dt.$$

You should find that the definite integral is repeated without alteration. This means that Mathcad has been unable to calculate an algebraic expression for an antiderivative of the integrand, and so it cannot evaluate symbolically the given definite integral.

(b) Evaluate the definite integral numerically, as follows. Click anywhere on the expression just created, and then enter `=`.

You should find that the answer 0.808 is displayed.

Comment

- ◇ To evaluate any definite integral numerically, you should enter it in the same way as for symbolic evaluation, then select it and enter `=`.
- ◇ The reason why some definite integrals can be evaluated numerically but not symbolically in Mathcad is that symbolic evaluation requires Mathcad to find an algebraic expression for an antiderivative, whereas numerical evaluation involves the use of a numerical algorithm. The answer obtained from this algorithm is an approximation, though usually an accurate one.

Mathcad notes

On rare occasions, the numerical method used by Mathcad for evaluating definite integrals fails to produce a value. In such a case, the integral is highlighted in red. Clicking on the integral reveals the error message 'This calculation does not converge to a solution.'.

Save the worksheet that you have created, if you wish. Then close the file.

The next activity requires the 'integration templates' Mathcad file that you used in the latter part of Subsection 5.1, but this time for application to definite integrals.

Activity 5.8 Further definite integrals

Open Mathcad file **221C2-01 Integration templates**, and turn to page 3 of the worksheet. This page can be used to evaluate the definite integral of $f(x)$ from a to b, for any function $f(x)$ and limits of integration a, b that you enter.

Note that three outputs are provided: for symbolic evaluation alone, for symbolic evaluation whose exact result is then evaluated as a decimal, and for direct numerical evaluation.

Evaluate each of the following integrals. (Enter the formula for each integrand into the definition of $f(x)$ in the template, enter the limits of integration into the definitions of a and b, and note the results.)

You may prefer to work in manual calculation mode here, to define f, a, b and then calculate the results. (See the first Mathcad note at the top of page 74.)

(a) $\displaystyle\int_0^1 \cos(\sin x)\,dx$ (b) $\displaystyle\int_0^{\pi} \cos(x^2)\,dx$ (c) $\displaystyle\int_0^{\pi} \cos^2 x\,dx$

Solutions are given on page 97.

If you find that Mathcad highlights a result in red for no apparent reason, or has not cleared a previous result, press `[Ctrl][F9]` to force a fresh calculation.

Comment

◇ Symbolic evaluation does not lead to a decimal answer in either of part (a) or (b), though direct numerical evaluation works in both cases. In part (a), Mathcad does not find an antiderivative of the integrand, and indicates this by repetition of the expression for the integral entered.

In part (b), an antiderivative expression *is* found, but it involves the obscure function 'FresnelC'. Attempting to evaluate this numerically causes the expression to be highlighted in red, and clicking on this expression reveals the error message 'This variable is undefined.'. The function FresnelC is one of a small number of functions that are recognised by Mathcad for symbolic purposes but which cannot be evaluated.

The Fresnel cosine integral function FresnelC(x) is defined to be

$$\int_0^x \cos\left(\tfrac{1}{2}\pi t^2\right)dt.$$

◇ The integral in part (c) can be evaluated either symbolically or by direct numerical evaluation. Symbolic evaluation gives $\frac{1}{2}\pi$.

In Mathcad many definite integrals can be evaluated either symbolically or by direct numerical evaluation. You might wonder which approach is more appropriate in any given situation. If you want an exact answer (so that you can see where constants such as π feature in it, for example), or if a general result is required, such as a formula for

$$\int_0^1 \cos(kx)\,dx \quad \text{(where } k \text{ is a non-zero constant),}$$

For example, the values of all the definite integrals in Activity 5.6 can be found satisfactorily using direct numerical evaluation.

then use the symbolic approach. If you simply want a decimal number that is an accurate value for the definite integral, then numerical evaluation should suffice. The definite integral template in Mathcad file 221C2-01 provides for all eventualities, giving where available both an exact value and a decimal value for the answer. When direct numerical evaluation and symbolic evaluation lead to the same decimal output, this is good evidence that the answers provided are correct.

This template can also be used to obtain symbolic expressions for definite integrals in which the interval endpoints are not given as specific numbers, provided that the definitions for a and b are either deleted or temporarily moved below the output results. For example, under these circumstances, the function $f(x) = \cos^2 x$ input for Activity 5.8(c) gives the output

$$\int_a^b f(x)\,dx \;\rightarrow\; \frac{b}{2} - \frac{a}{2} - \frac{\sin(2a)}{4} + \frac{\sin(2b)}{4}.$$

However, neither symbolic nor direct numerical evaluation can be guaranteed always to give an accurate answer, as the following activity shows.

Activity 5.9 Mathcad may not give the right answer!

You should still be working with Mathcad file 221C2-01, on page 3 of the worksheet.

In part (a), you may need to interrupt calculation with the `[Esc]` key, to stop Mathcad seeking a symbolic solution.

In each case below, try to evaluate the definite integral given by hand. Then use Mathcad to try and evaluate the given integral.

(a) $\displaystyle\int_0^{\pi/4} \sec^4 x\,\sqrt{\tan x}\,dx$

(b) $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$

(c) $\displaystyle\int_{10^{-5}}^{1} \frac{1}{x^2}\cos\left(\frac{1}{x}\right)\,dx$

Comment

(a) Using the substitution $u = \tan x$, we find that

$$\begin{aligned}\int_0^{\pi/4} \sec^4 x\,\sqrt{\tan x}\,dx &= \int_0^{\pi/4} \sec^2 x\,(1+\tan^2 x)\sqrt{\tan x}\,dx\\ &= \int_0^1 (u^{1/2} + u^{5/2})\,du\\ &= \left[\tfrac{2}{3}u^{3/2} + \tfrac{2}{7}u^{7/2}\right]_0^1\\ &= \tfrac{2}{3} + \tfrac{2}{7} = \tfrac{20}{21} = 0.952 \quad \text{(to 3 d.p.).}\end{aligned}$$

Mathcad fails to find an antiderivative in this case, though direct numerical evaluation gives the answer correct to three decimal places.

(b) The following ‘calculation’ is *not* correct!

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -1 - (-(-1)) = -2.$$

The integrand $1/x^2$ is not defined for $x = 0$, and so we should not attempt to integrate it over any interval, such as $[-1, 1]$, which contains 0.

However, Mathcad does supply answers! Symbolic evaluation gives the answer ∞, which provides some indication that there is a difficulty in evaluating the integral. The wrong answer from direct numerical evaluation, 1.376×10^3, provides no such warning.

(c) Using the substitution $u = 1/x$, we find that

$$\begin{aligned}\int_{10^{-5}}^{1} \frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx &= -\int_{10^5}^{1} \cos u\, du \\ &= \int_{1}^{10^5} \cos u\, du \\ &= [\sin u]_1^{10^5} \\ &= \sin(10^5) - \sin 1 \simeq -0.806.\end{aligned}$$

Symbolic evaluation in Mathcad agrees with this answer, while direct numerical evaluation fails to produce a result. (The problem here is that the integrand given has a large number of oscillations packed into the left-hand end of the interval of integration.)

Mathcad notes

The infinity symbol ∞ can be obtained from a button on the 'Calculus' toolbar, or by typing `[Ctrl][Shift]z`. If you evaluate it numerically, then Mathcad gives the value 1×10^{307}, which is the largest power of 10 that Mathcad can handle.

Now close Mathcad file 221C2-01.

Usually, Mathcad gives accurate and helpful responses when asked to integrate. However, you have just seen some examples which show that Mathcad's results sometimes need to be treated with caution. Just occasionally, its numerical calculations may give an inaccurate answer. Sometimes, its symbolic responses are not totally correct or are in a form that is not helpful.

Similar remarks apply to *any* numerical or symbolic manipulation package.

The broad message to take away from this experience is that any results obtained from Mathcad should be treated critically. For example, when you evaluate an integral numerically, you should try to estimate the answer by alternative means, so that you have something against which to compare the Mathcad result.

You also saw, in Activity 5.9(b), that hand calculation sometimes needs to be approached with care. Watch out for points in the proposed interval of integration at which the integrand is undefined, and do not integrate a function over any interval in which the function is not continuous.

5.3 Volumes of solids of revolution (Optional)

In this subsection you can use a prepared Mathcad file to explore the volumes of solids of revolution. The solid of revolution generated by the region bounded by the graph of a continuous function f from a to b is illustrated in Figure 5.1.

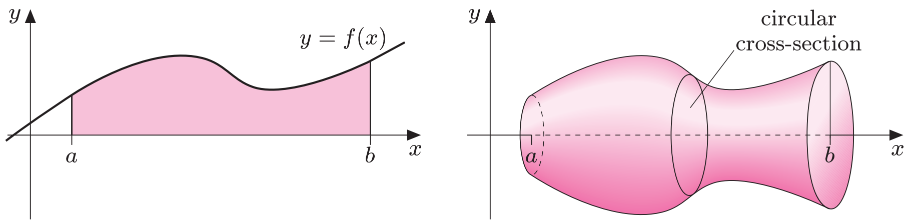


Figure 5.1 Volume of revolution

See Chapter C2, Section 4.

In the main text you saw a derivation of the following result.

Volume of solids of revolution

If f is any function that is continuous on $[a, b]$, then the volume of the solid of revolution generated by the region bounded by the graph of f from $x = a$ to $x = b$ is given by

$$\text{volume of revolution} = \pi \int_a^b (f(x))^2 \, dx. \tag{5.2}$$

You also saw that the volume of a solid of revolution can be approximated by summing the volumes of a set of cylinders. Figure 5.2 shows a set of N cylinders whose total volume approximates the volume of the solid in Figure 5.1.

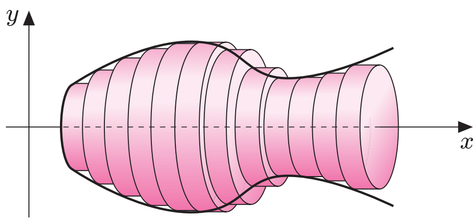


Figure 5.2 ‘Sum of cylinders’

Each cylinder has ‘width’ h, where $h = (b - a)/N$, and thus has its left-hand edge at the point $x = a + ih$, where the value of i is 0 for the left-most cylinder, 1 for the next cylinder, and so on, up to $N - 1$ for the final cylinder. For $i = 0, 1, \ldots, N - 1$, the radius of the ith cylinder is $f(a + ih)$. The total volume of the cylinders is therefore given by

$$\pi h \sum_{i=0}^{N-1} (f(a + ih))^2, \quad \text{where } h = (b - a)/N. \tag{5.3}$$

In general, the greater the number N of cylinders, the more closely sum (5.3) approximates the volume of the solid.

In the following, *optional*, activity you can use a prepared Mathcad file to explore such approximations for volumes of solids of revolution.

Activity 5.10 Volumes of solids of revolution (Optional)

Open Mathcad file **221C2-02 Volumes of solids of revolution**. The worksheet consists of a single page.

A function f, constants a and b, and a number N of subintervals are defined near the top of the page. The volume of the solid of revolution generated by rotating the region bounded by the graph of f between $x = a$ and $x = b$ is estimated by summing the volumes of the corresponding N cylinders, and is also calculated by evaluating the definite integral in equation (5.2). Both values are displayed near the middle of the page.

A graph illustrates the solid of revolution. It shows the 'outline' of the solid when the variable C is 0, and it shows the N cylinders if C is set to 1.

The cylinders are not drawn if N is greater than 20 because there is a limit to the number of points that Mathcad can plot on a graph.

(a) Initially $f(x) = \frac{1}{4}x$, $a = 0$ and $b = 4$, so the solid is a right circular cone with height 4 and base radius 1. The number of cylinders is $N = 4$.

(i) Set $C = 1$ to see the cylinders. You will find that only three of the four cylinders are visible because the radius of the first is zero.

(ii) Set N to 5, 10, 100 and 1000 in turn, and observe the effect on the estimate obtained by summing the volumes of the cylinders.

You may also like to try $N = 20$, to see the maximum number of cylinders that the worksheet can display.

(iii) Set $f(x) = 1 - \frac{1}{4}x$ and repeat part (a)(ii).

(b) Set $f(x) = \sin x$ and $b = \pi$; keep $a = 0$. Try varying the value of N, and observe the effect on the estimate.

(c) You may like to use the worksheet to explore, and to check the answers for, some of the other examples given in Section 4 of the main text. For example, if you would like to explore the volume of a sphere of radius 1, then set $f(x) = \sqrt{1 - x^2}$, $a = -1$ and $b = 1$. You will have to resize the graph if you want the sphere to look spherical.

The volume of a sphere of radius r was determined in Activity 4.1 in the main text.

You may also like to try the function $f(x) = 1 + \frac{1}{2}\sin x$ over the interval $[0, 6]$, which gives a pleasing vase shape! You can alter the shape of the vase by varying the definition of f.

Comment

(a) In each of parts (ii) and (iii) the solid is a right circular cone with height 4 and base radius 1. So the volume is $\frac{4}{3}\pi \simeq 4.189$. In both cases, the estimate gets closer to this value of the definite integral as N is increased, as expected. In part (ii) the estimate is always less than the value of the definite integral, whereas in part (iii) it is always greater.

This volume was found using the formula from Example 4.1 in the main text.

(b) The volume of the solid in this case is $\frac{1}{2}\pi^2 \simeq 4.935$. You may have been surprised to find that the estimate always appears to be *equal* to the value of the definite integral in this case (except when $N = 1$). This is actually true, and the proof is not hard, but it would take too much space to give it here.

This volume was found in Activity 4.2(b) in the main text.

Mathcad notes

- ◇ The summation sign is obtained from the $\sum_{n=1}^{m}$ button on the 'Calculus' toolbar, or by typing `[Ctrl]$` (for which you have to press the three keys `[Ctrl]`, `[Shift]` and 4 together).
- ◇ The cylinders are filled in by plotting zigzag lines very close together. Tiny gaps between the lines may appear if the graph is printed.

Now close Mathcad file 221C2-02.

Chapter C3, Section 5
Taylor series with the computer

Mathcad can be used both to find a given number of terms of a Taylor series and to show how the graphs of the resulting Taylor polynomials compare with that of the original function. In Subsection 5.1 you will see how to find Taylor polynomials for a given function about a specified point, while the graphical approach in Subsection 5.2 illustrates how these polynomials approximate the given function.

The single Mathcad file for this section includes both symbolic and graphical templates for the investigation of Taylor polynomials.

5.1 Using Mathcad to find Taylor series

In this subsection you will learn how to use Mathcad to find the first few terms of a Taylor series.

Activity 5.1 Taylor series in Mathcad

Open Mathcad file **221C3-01 Taylor series and polynomials**. Page 1 introduces the worksheet. Work through page 2, and then turn to page 3.

Page 2 describes how to enter the Mathcad instructions for finding a Taylor polynomial. Rather than expecting you to do this from scratch on each occasion, page 3 of the worksheet provides a 'Taylor series template' which reduces the amount of work involved. This page can be used to find the Taylor polynomial for any function $f(x)$ that you input, provided that the centre a and number n of terms in the polynomial are also specified. The template is set up with $f(x) = e^x$, $a = 0$ and $n = 6$, giving the same Taylor polynomial that you found on page 2 of the worksheet.

The first four terms agree with the quartic Taylor polynomial for $f(x) = e^x$, found in Example 2.1. The polynomial here also agrees with the first five terms of the Taylor series found in Example 3.1.

Each of parts (a)–(c) below gives a standard function, together with its Taylor series about 0. In each case, enter the formula for the function into the definition of $f(x)$ in the template, choose $n = 9$, and confirm that the polynomial output from Mathcad agrees with the corresponding Taylor series for each term shown on screen.

These series are stated in Subsection 3.2, in the table on page 32 of the main text.

(a) $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots$

(b) $\sin x = x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5 - \dfrac{1}{7!}x^7 + \dfrac{1}{9!}x^9 - \cdots$

(c) $\cos x = 1 - \dfrac{1}{2!}x^2 + \dfrac{1}{4!}x^4 - \dfrac{1}{6!}x^6 + \dfrac{1}{8!}x^8 - \cdots$

Comment

◇ The Mathcad output agrees in each case with the given series. For parts (b) and (c), this follows because

$$3! = 6, \quad 4! = 24, \quad 5! = 120, \quad 6! = 720,$$
$$7! = 5040, \quad 8! = 40\,320 \quad 9! = 362\,880.$$

◇ As noted in the main text, $\sin x$ is an odd function, and hence its Taylor series has no term involving an even power of x. Similarly, $\cos x$ is an even function, and so its Taylor series has no term involving an odd power of x.

Mathcad notes

- For a series about 0, Mathcad displays the non-zero terms from x^k to x^{n+k-1}, where the first non-zero term involves x^k. Suppose as above that $n = 9$. With $k = 0$ (as for e^x and $\cos x$), the highest power shown is $x^{n+k-1} = x^8$, but with $k = 1$ (as for $\ln(1+x)$ and $\sin x$), the highest power shown is x^9. For $x^2 \ln(1+x)$, where $k = 3$, the highest power shown is x^{11}.
- Mathcad's symbolic keyword 'series' can be used to find Taylor polynomials up to 100 non-zero terms (e.g. up to the x^{100} term for e^x, but up to x^{198} for $\cos x$ and up to x^{199} for $\sin x$).

For some Taylor polynomials, Mathcad cannot display this many terms.

In the next activity, you are invited to use Mathcad to find Taylor polynomials in further instances where corresponding results were obtained by hand in the main text.

Activity 5.2 Further Taylor polynomials

For each of parts (a)–(d) below, use the worksheet to find the Taylor polynomial specified, and compare your answer with that obtained earlier in the main text and stated again here. (You will need to specify a and n in the worksheet, as well as $f(x)$.)

You should still be working with Mathcad file 221C3-01, on page 3 of the worksheet.

(a) $f(x) = \ln x$, about $x = 1$, up to the term in $(x-1)^4$

$p(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$

See Example 2.2.

(b) $f(x) = \sin x$, about $x = \frac{1}{6}\pi$, up to the term in $(x - \frac{1}{6}\pi)^3$

$p(x) = \frac{1}{2} + \frac{1}{2}\sqrt{3}(x - \frac{1}{6}\pi) - \frac{1}{4}(x - \frac{1}{6}\pi)^2 - \frac{1}{12}\sqrt{3}(x - \frac{1}{6}\pi)^3$

See Activity 2.3.

(c) $f(x) = e^x \cos x$, about $x = 0$, up to the term in x^3

$p(x) = 1 + x - \frac{1}{3}x^3$

See Example 4.4.

(d) $f(x) = \arcsin x$, about $x = 0$, up to the term in x^5

$p(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5$

See Activity 4.10(c). Recall that arcsin should be entered as 'asin' in Mathcad.

Comment

The Mathcad answer agrees in each case with that in the main text.

You have seen that many Taylor series can be expressed concisely using sigma notation. For example, the Taylor series about 0 for the exponential function can be written concisely as

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k, \quad \text{for } x \in \mathbb{R}.$$

Similarly, the Taylor series about 0 in Activity 5.1 can be expressed as

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k, \quad \text{for } -1 < x < 1;$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \quad \text{for } x \in \mathbb{R};$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}, \quad \text{for } x \in \mathbb{R}.$$

These expressions for $\sin x$ and $\cos x$ were given in Subsection 3.1, on page 32 of the main text.

In the next activity you can use Mathcad to find the first few terms of two Taylor series. You are asked to look for patterns in these terms, to help you in conjecturing expressions in sigma notation for the series.

Activity 5.3 Taylor series in sigma notation

You should still be working with Mathcad file 221C3-01, on page 3 of the worksheet.

For each of parts (a) and (b) below, use the worksheet to find the Taylor series about 0 for the given function, up to the term in x^5. Then conjecture the next three terms of the series, and use Mathcad to verify your answer. Finally, try to conjecture the whole series, and write down your answer using sigma notation.

(a) $f(x) = \dfrac{2-x}{(1-x)^2}$ (b) $f(x) = \dfrac{x^2-x}{(1+x)^3}$

Solutions are given on page 97.

Comment

The Taylor series for $1/(1+x)^2$ was found in Activity 4.4(a) in the main text.

◇ You could have calculated the first few terms of these Taylor series by hand, though this would have been much slower than using the computer. For example, for part (a) you can use the binomial series to find the Taylor series about 0 for $1/(1+x)^2$, and then substitute $-x$ for x to find the Taylor series about 0 for $1/(1-x)^2$. Multiplying this series by the polynomial $2-x$ then gives the required Taylor series.

◇ You can use Mathcad to check your final answers, if you wish. First create some extra space below the final line on page 3 of the worksheet, by placing the red cross cursor there and pressing `[Enter]` to insert some blank lines. Then enter the summation sign by clicking on the $\sum_{n=1}^{m}$ button on the 'Calculus' toolbar, or by typing `[Ctrl]$` (for which you have to press the three keys `[Ctrl]`, `[Shift]` and `4` together). Enter your expression for the kth term of the Taylor series. On the summation sign enter k and the lower limit into the bottom two placeholders, and the symbol ∞ into the top placeholder (either from a button on the 'Calculus' toolbar or by typing `[Ctrl][Shift]z`).

The `[Tab]` key provides a quick way of moving round the placeholders on the summation sign.

Note that Mathcad uses the standard exclamation mark notation for factorials. This is available from the $n!$ button on the 'Calculator' toolbar or by typing `!` (given by `[Shift]1`).

To evaluate this expression symbolically, click on the → button on the 'Symbolic' toolbar, or use `[Ctrl].`, the keyboard alternative. Then click elsewhere on the page, or press `[Enter]`. If your expression in sigma notation is correct, then Mathcad should respond with an answer that is equivalent to the original expression given in the activity. (If this check does not work, then try replacing ∞ by n in the top placeholder, to check the first few terms of the series.)

This part (a) output takes a long time to emerge.

Alternatively, use the 'assume' button on the 'Symbolic' toolbar.

The answer to part (b) is obtained rapidly in this manner. The output for part (a) contains an unevaluated limit (whose value is zero for $|x| < 1$, which is the range of validity for the series). The result for part (a) can be obtained explicitly by using the 'assume' keyword, and typing (after the expression for the summation)

```
[Ctrl][Shift].assume,x[Ctrl]=RealRange(-1,1)
```

before evaluating symbolically.

5.2 Graphs of Taylor polynomials

In this subsection, you will see how the graphs of Taylor polynomials approximate that of the original function more and more closely as the degree of the polynomial is increased. By observing this behaviour, it is possible to estimate the 'maximum' ranges of validity of the corresponding Taylor series.

Activity 5.4 Exploring graphs of Taylor polynomials

Turn to page 4 of the Mathcad worksheet. This page is designed for plotting the graphs of a function $f(x)$ (in black) and its Taylor polynomial of degree n about a (in red).

You should still be working with Mathcad file 221C3-01.

(a) The page is set up initially to plot the function $f(x) = e^x$ and its Taylor polynomial of degree 0 about 0.

(i) Increase the value of n to 1, 2, 3 and 4 in turn, and observe the effect on the graph of the Taylor polynomial.

You can see the graphs of all these Taylor polynomials at the same time if you set $m = 5$. Try this now, and reset $m = 1$ afterwards.

You can set m to any integer value between 1 and 5, to display graphs of the m Taylor polynomials of degrees $n - m + 1$ up to n.

(ii) Set the x- and y-axis limits as follows: $X1 = -10$, $X2 = 5$, $Y1 = -5$ and $Y2 = 40$.

(iii) Notice from the graph that the current Taylor polynomial (of degree 4) appears to approximate f closely over an interval which is approximately $[-1.5, 1.5]$.

The perception of this interval depends to some extent on the vertical scale of the graph.

Increase n to 10, 20 and 30 in turn, and observe the effect on the interval over which the polynomial appears to approximate f closely.

(b) Now explore the Taylor polynomials about 0 of the function $f(x) = 1/(1 - x)$, by following the instructions below.

(i) Reset $n = 0$, then enter the expression $1/(1 - x)$ into the definition of $f(x)$.

(ii) Set the x- and y-axis limits as follows: $X1 = -1.5$, $X2 = 2.5$, $Y1 = -5$ and $Y2 = 20$.

(iii) Increase the value of n to 1, 2, 3, 4, 10, 20 and 30 in turn, and observe the effect on the interval over which the polynomial appears to approximate f closely.

Comment

◇ In part (a), as n increases, the left-hand endpoint of the interval over which the Taylor polynomial appears to approximate the function closely seems to move without limit in the negative direction. It is more difficult to see what happens to the right-hand endpoint, as the graph of the function is steep to the right of the y-axis, but in fact it moves without limit also in the positive direction.

In part (b), as n increases, the left-hand and right-hand endpoints seem to tend to -1 and 1, respectively.

These observations are explained by the facts that the Taylor series about 0 for e^x is valid for all $x \in \mathbb{R}$, whereas the Taylor series about 0 for $1/(1 - x)$ is valid only for x in the interval $(-1, 1)$.

◇ The current Mathcad page includes a feature that can be used for checking calculations in which Taylor polynomials are used to find approximate values of functions at particular points. You carried out calculations of this type in Subsection 2.3 of the main text.

If you set $n = 5$ and $m = 5$ and scroll down to the bottom of the page, then you will see that the Taylor polynomials of degrees 1, 2, 3, 4 and 5 for f about a are evaluated at a particular value of x, namely, $x = 0.25$. As expected, as n increases, these values become progressively closer to the value of $f(x)$, which is also displayed. The corresponding remainders are also shown in a table. You can change the value of x here to find approximations for f at a different point,

and you can increase the value of n to calculate approximations using higher-degree Taylor polynomials. The table contains the values given by the m Taylor polynomials of degrees $n - m + 1$ up to n.

Mathcad notes

If you try to plot the graph of a function f using a range variable x, and the formula for the rule of f is not defined at one or more of the values of x, then Mathcad usually avoids the problem by omitting any such value from the range. For example, in part (b) Mathcad is able to plot the graph of the function $f(x) = 1/(1-x)$, although the formula gives no value for $x = 1$.

However, sometimes rounding errors in Mathcad's internal calculations may alter slightly the values taken by the range variable and cause a spurious vertical line to appear at the 'problem value'. This happens because Mathcad graphs are produced by plotting a point for each value taken by the range variable and then joining successive points with line segments. If the values taken by the function are large and positive for x-values on one side of the problem value, and are large and negative on the other side, then the graph will include an apparently vertical line joining the two 'branches' of the graph. (In this worksheet the step size used for the graph range is set to a power of 2; for example, a step size of $\frac{1}{128}$ $(= 2^{-7})$ is used rather than $\frac{1}{100}$. This reduces the likelihood of rounding errors, because such numbers can be stored and manipulated very accurately in binary form inside the computer.)

The next activity is similar to Activity 5.4, but it involves even and odd functions.

Activity 5.5 Taylor polynomials of even and odd functions

You should still be working with Mathcad file 221C3-01, on page 4 of the worksheet.

In this activity you will explore the graphs of Taylor polynomials of even and odd functions.

For each of parts (a)–(c) below, explore the Taylor polynomials about 0 of the function $f(x)$ given, as follows.

(i) Reset $n = 0$, then enter the expression given into the definition of $f(x)$.

(ii) Set the x-axis limits $X1$, $X2$ and the y-axis limits $Y1$, $Y2$ to the values indicated.

(iii) Increase the value of n to 1, 2, 3, 4, 10, 20 and 30 in turn, and observe the effect on the interval over which the polynomial appears to approximate f closely.

(a) $f(x) = \cos x; \quad X1 = -10,\ X2 = 10,\ Y1 = -1.5,\ Y2 = 1.5$

(b) $f(x) = \sin x; \quad X1 = -10,\ X2 = 10,\ Y1 = -1.5,\ Y2 = 1.5$

You found the Taylor series about 0 for this function in Example 4.1.

(c) $f(x) = \dfrac{1}{1+x^2}; \quad X1 = -1.5,\ X2 = 1.5,\ Y1 = -0.5,\ Y2 = 1.5$

Comment

(a) The 'function(f) =' statement in the worksheet now shows "is EVEN" on the right-hand side, whereas in Activity 5.4 the right-hand side was "is neither even nor odd". The function is even because $f(x) = f(-x)$, and the graphs of both the function and its Taylor polynomials are symmetric about the y-axis (unchanged under reflection in the y-axis).

As n increases, the left-hand and right-hand endpoints of the interval over which the Taylor polynomial appears to approximate the cosine function closely seem to move without limit in the negative and positive directions, respectively.

The graphs of the Taylor polynomials of degrees 1 and 3 about 0 are the same as those of degrees 0 and 2, respectively; this is as expected, since the cosine function is even.

(b) The ‘function(f) =’ statement in the worksheet now shows “is ODD” on the right-hand side. The function is odd because $f(x) = -f(-x)$, and the graphs of both the function and its Taylor polynomials are unchanged by rotation through the angle π about the origin.

As n increases, the left-hand and right-hand endpoints of the interval over which the Taylor polynomial appears to approximate the sine function closely seem to move without limit in the negative and positive directions, respectively.

The graphs of the Taylor polynomials of degrees 2 and 4 about 0 are the same as those of degrees 1 and 3, respectively, as expected, since the sine function is odd.

(c) The function $f(x) = 1/(1+x^2)$ is even. The left-hand and right-hand endpoints of the interval over which the Taylor polynomial appears to approximate this function closely tend to -1 and 1, respectively.

The observations about the endpoints of the intervals in parts (a), (b) and (c) are explained by the facts that the Taylor series about 0 for $\cos x$ and $\sin x$ are both valid for all $x \in \mathbb{R}$, whereas the Taylor series about 0 for $1/(1+x^2)$ is valid only for values of x in the interval $(-1, 1)$.

If the function is even or odd, then the effect of setting m to a value other than 1 differs from that in Activity 5.4. The graphs now plotted are those for the m Taylor polynomials of degrees $n - 2m + 2$ up to n, at intervals of 2. However, m graphs are displayed (for large enough n) in either case.

In the final activity of this section you will explore two Taylor series that you have not met before in MS221. You are asked to use the Mathcad worksheet to estimate their ‘maximum’ ranges of validity.

Activity 5.6 Exploring ranges of validity

For each of the functions in parts (a) and (b) below, decide whether the function is even, odd, or neither. By choosing appropriate values for the x- and y-axis limits, and then increasing the value of n from 0, estimate the ‘maximum’ range of validity of the Taylor series about 0 for the function.

You should still be working with Mathcad file 221C3-01, on page 4 of the worksheet.

(a) $f(x) = \dfrac{1}{3-x}$ (b) $f(x) = \dfrac{x}{1+4x^2}$

Solutions are given on page 97.

Mathcad notes

You may find that, for Taylor polynomials of higher degree, no graph of the polynomial appears (nor any error message). In this case, try again after increasing the value of $X1$ or reducing the value of $X2$.

Now close Mathcad file 221C3-01.

Solutions to Activities

Chapter C1

Solution 5.1

Where more than one expression is given for a solution here, the first is similar to the Mathcad output and the second is a form that you are more likely to obtain by hand.

(a) $3x^2 - 12x - 15$

(b) $4\pi r^2$

(c) $e^{-t}(\cos(t) - 2t) - e^{-t}\left(\sin(t) - t^2\right)$
$$= \frac{\cos t - \sin t + t^2 - 2t}{e^t}$$

(d) $2x\cos(x^2)$

(e) $-4\sin(4x)$

(f) $\dfrac{t^2+3}{t} + 2t\ln(t)$

(g) $\dfrac{1}{u(u^2+3)} - \dfrac{2u\ln(u)}{(u^2+3)^2} = \dfrac{u^2+3-2u^2\ln u}{u(u^2+3)^2}$

(h) $\dfrac{e^t+1}{t+e^t}$

Solution 5.3

(a) Mathcad ('simplify' or 'factor') supplies the answer $e^{x^2}(1+2x^2)$, which is equivalent to the stated answer.

(b) Mathcad ('factor') supplies the answer
$$\frac{\sqrt{x}\cos\left(\sqrt{x}\right) - \sin\left(\sqrt{x}\right)}{2x^{3/2}},$$
which is the same as the stated answer.

(c) Recalling that $1/\cos x = \sec x$, Mathcad ($\to$) supplies an answer equivalent to
$$-\sin[(x+4)\sec x][\sec x + (x+4)\sec x\tan x],$$
which is in turn equivalent to the stated answer, since
$$\sec x + (x+4)\sec x\tan x = \sec x(1+(x+4)\tan x).$$

(d) Mathcad ('simplify') supplies the answer
$$x^3e^x(4\sin x + x\cos x + x\sin x).$$
Since we have
$$4\sin x + x\sin x = (4+x)\sin x,$$
this is equivalent to the stated answer.

(e) Mathcad ('simplify' or 'factor') supplies an answer equivalent to
$$\frac{x^3 + x + x^3\tan^2 x + x\tan^2 x - x^2\tan x + \tan x}{(1+x^2)^2}.$$
The numerator here is
$$\begin{aligned}&x^3 + x + x^3\tan^2 x + x\tan^2 x - x^2\tan x + \tan x\\ &= (x^3+x)(1+\tan^2 x) + (1-x^2)\tan x\\ &= x(1+x^2)\sec^2 x + (1-x^2)\tan x\end{aligned}$$
(since $1 + \tan^2 x = \sec^2 x$), showing that the Mathcad output is equivalent to the stated answer.

Solution 5.4

(a) In part (iii), the table confirms that $f(x_4) = 0$ to nine decimal places, so $x_4 = 1.769\,292\,354$ is a good approximation for the zero of f.

(The fact that $f(x_4) = 0$ to nine decimal places does not, however, guarantee that x_4 is an approximation for the zero accurate to nine decimal places, although this is often the case.)

(b) (i) With $x_0 = 0.82$, the tangent in the first iteration is nearly horizontal because 0.82 is close to a stationary point of f. This causes the tangent to cross the x-axis at a point far away from the zero that we seek, and so the next term x_1 is large. In fact, from the table, $x_1 = 180$ to the nearest integer.

(ii) The smallest value of n with $f(x_n) = 0$ to nine decimal places is $n = 17$.

(c) With $x_0 = 0$, the terms of the sequence alternate between 0 and -1; that is, these two values form a 2-cycle. So with this starting value, the sequence never approaches the zero of f.

Solution 5.5

(a) There is one solution in the interval $[1,2]$ (namely $\frac{1}{2}(1+\sqrt{5})$), whose value is given as $1.618\,033\,989$. (The other solution of the quadratic equation lies outside the given interval.)

(b) There are three solutions in the interval $[-3,3]$, whose values are given as $-2.086\,130\,198$, $0.571\,993\,268$ and $2.514\,136\,929$.

(c) There is one solution in the interval $[-1,0]$, whose value is given as $-0.567\,143\,290$.

Chapter C2

Solution 5.2

One difference in each case is that the Mathcad answer does not include an arbitrary constant.

(a) The answer supplied by Mathcad is

$$\frac{2e^{7u}}{7} + \frac{u^2}{2},$$

which is equivalent to the given answer.

(b) The Mathcad answer is

$$\frac{x}{2} + \frac{\sin(2x)}{4},$$

which is equivalent to the given answer.

(c) The Mathcad answer is

$$\frac{\ln(x^3+1)}{3}.$$

The two answers differ in that the Mathcad answer has brackets around $x^3 + 1$, whereas there are modulus signs in the given answer. The Mathcad answer is thus valid only for a restricted set of values of x, namely, $1 + x^3 > 0$.

Solution 5.6

You should have obtained the following Mathcad answers, in which the numerical values are given to 3 decimal places.

(a) $\dfrac{2}{\pi} = 0.637$

(b) 2

(c) $\dfrac{3e^4}{4} - \dfrac{e^2}{4} = 39.101$

(d) $\dfrac{\pi\sqrt{2}}{72} - \dfrac{\sqrt{2}}{18} + \dfrac{1}{9} = 0.094$

The exact answers given by Mathcad agree with those in the main text, though there is some rearrangement. The numerical answers given by Mathcad also agree with those in the main text, at least to the number of decimal places that the answers have in common. To increase the number of decimal places given by Mathcad, it is necessary to change the value of 'Number of decimal places', as described in the first Mathcad note for Activity 5.5, on page 83 of this computer book.

Solution 5.8

(a) 0.869

(b) 0.566

(c) 1.571

Chapter C3

Solution 5.3

(a) Up to the term in x^5, the Taylor series about 0 is

$$2 + 3x + 4x^2 + 5x^3 + 6x^4 + 7x^5 + \cdots.$$

The part of the series consisting of the next three terms is

$$\cdots + 8x^6 + 9x^7 + 10x^8 + \cdots.$$

In each term of the series the coefficient is two more than the power of x, so the whole series is

$$\sum_{k=0}^{\infty} (k+2)x^k.$$

(b) Up to the term in x^5, the Taylor series about 0 is

$$-x + 4x^2 - 9x^3 + 16x^4 - 25x^5 + \cdots.$$

The part of the series consisting of the next three terms is

$$\cdots + 36x^6 - 49x^7 + 64x^8 + \cdots.$$

In each term of the series the coefficient is the square of the power of x, multiplied by -1 when the power is odd, so the whole series is

$$\sum_{k=0}^{\infty} (-1)^k k^2 x^k.$$

(The lower limit of the sum here could be taken to be 1 instead of 0, since the term obtained by taking $k = 0$ is 0.)

Solution 5.6

(a) The function is neither even nor odd. Reasonable values for the x- and y-axis limits in this case are $X1 = -5$, $X2 = 5$, $Y1 = -5$, $Y2 = 5$. It appears that the maximum range of validity is $-3 < x < 3$.

(b) The function is odd. Reasonable values for the x- and y-axis limits in this case are $X1 = -0.8$, $X2 = 0.8$, $Y1 = -1$, $Y2 = 1$. It appears that the maximum range of validity is $-\frac{1}{2} < x < \frac{1}{2}$.

Computer Book D

Structure in Mathematics

Guidance notes

This computer book contains those sections of the chapters in Block D which require you to use Mathcad. Each of these chapters contains instructions as to when you should first refer to particular material in this computer book, so you are advised not to work on the activities here until you have reached the appropriate points in the chapters.

In order to use this computer book, you will need the following Mathcad files.

Chapter D1

221D1-01 Cartesian and polar forms
221D1-02 Roots of unity
221D1-03 Complex exponentials (Optional)

Chapter D2

221D2-01 Remainders of powers
221D2-02 Modular arithmetic

Chapter D3

221D3-01 Piecewise linear paths (Optional)
221D3-02 PLPs with symmetry (Optional)
221D3-03 A rose window (Optional)

Instructions for installing these files onto your computer's hard disk, and for opening them, are given in Chapter A0 of MST121.

The computer activities for Chapters D1 and D2 also require you to work with Mathcad worksheets which you have created yourself.

Activities based on software vary both in nature and in length. Sometimes the instructions for an activity appear only in the computer book; in other cases, instructions are given in the computer book and on screen. Feedback on an activity is sometimes provided on screen and sometimes given in the computer book.

For advice on how each computer session fits into suggested study patterns, refer to the Study guides in the chapters.

Note that there are no specified computer activities associated with Chapter D4.

Chapter D1, Section 6
Complex numbers and Mathcad

6.1 Complex arithmetic in Mathcad

This symbolic keyword was introduced in MST121 Chapter A3; see also *A Guide to Mathcad*.

Certain quadratic equations have roots that are complex numbers, and you may have already seen Mathcad produce such roots in this context. For example, suppose that you use the symbolic keyword 'solve' to solve the equation $x^2 - 6x + 25 = 0$. Mathcad gives the two solutions as

$$\begin{pmatrix} 3+4i \\ 3-4i \end{pmatrix}.$$

This is as we would expect, since the equation has the two solutions $3 \pm 4i$. Note that Mathcad uses the usual symbol i for $\sqrt{-1}$.

Mathcad will perform various manipulations with complex numbers. To use it for this purpose, you need first to be able to enter complex numbers into Mathcad. (This requires a little care, since Mathcad needs to know when the symbol i is being used to mean $\sqrt{-1}$, rather than in some other way, such as, say, part of a variable name.) For example, to enter $3+4i$, you can click on the buttons on the 'Calculator' toolbar, including the i button, or just type `3+4i`. (You do *not* enter a multiplication between the `4` and the `i`.) However, when i is entered alone in Mathcad, it must be entered as $1i$. This extra '1' in front of the i is entered for you automatically if you click on the i button, but when using the keyboard, you must type `1i`. For example, to enter $2-i$ via the keyboard, you type `2-1i`. (The '1' is visible only when entering or editing the complex number. It disappears once the number is complete.)

Activity 6.1 Entering complex numbers in Mathcad

(a) By hand, simplify each of the following complex numbers.

(i) $(3+4i)(3-4i)$ (ii) $i(3-4i)$

(b) Create a new (Normal) worksheet. Then enter each of the complex products in part (a), and enter `=` to evaluate them.

Comment

(a) These products are

(i) 25; (ii) $4+3i$.

(b) Mathcad gives the above values. If you use keyboard entry, and type `i` or `1*i` to enter i in (ii), then Mathcad gives an error, treating i as an undefined variable, not as $\sqrt{-1}$. The key sequence required here is `1i*(3-4i)`. (Whatever entry method you use for (ii), you *must* enter the multiplication between the i and the left bracket – Mathcad will *not* insert it if you omit to do so.)

The next activity provides further practice in entering complex numbers.

Activity 6.2 Multiplying complex numbers

(a) By hand, evaluate $(1-2i)(3+4i)$.

(b) On a new (Normal) worksheet, enter the expressions shown below. For each of zw and $(ac-bd)+(ad+bc)i$, click on the expression and then enter `=` to evaluate it. Experiment with other values of a, b, c and d, if you wish.

```
a := 1      b := -2      c := 3      d := 4

z := a + bi              w := c + di

z w

(a c - b d) + (a d + b c)i
```

To enter $a+bi$, use the buttons on the 'Calculator' toolbar or type `a+b*1i`. (Note that, when variables are used to construct a complex number, a multiplication `*` is required before the i, which must be typed as `1i`.)

Comment

(a) This product is $11-2i$, as Mathcad confirms.

(b) The two expressions give the same value whatever values are specified for a, b, c and d, as one would expect!

Leave this worksheet open; it will be used again for the next activity.

Mathcad can evaluate the real and imaginary parts of a complex number, complex conjugates and the modulus of a complex number. It uses the usual notations: $\text{Re}(z)$ for the real part of z, $\text{Im}(z)$ for the imaginary part of z, $\overline{z}$ for the complex conjugate of z, and $|z|$ for the modulus of z. To enter the first two, one just types the expressions `Re(z)` or `Im(z)`. To enter $\overline{z}$, type `z[Shift]2`. To enter $|z|$, type `z`, then click on the $|x|$ button on the 'Calculator' toolbar or use the keyboard alternative `[Shift]\`.

Remember that names are case sensitive in Mathcad. You *must* use capital `R` and `I` here.

Activity 6.3 Complex arithmetic

(a) With the complex numbers $z=1-2i$ and $w=3+4i$ entered on a worksheet, as you did for the previous activity, use Mathcad to calculate each of (i)–(ix) below.

(i) $z+w$ (ii) $w+z$ (iii) $\text{Re}(z)$ (iv) $\text{Im}(w)$ (v) $\overline{z}$

(vi) $|w|$ (vii) $1/w$ (viii) w/z (ix) $w\overline{z}/|z|^2$

(b) Try varying a, b, c and d, and check that (viii) and (ix) in part (a) give the same answers each time.

Comment

(a) Mathcad gives the following answers.

(i) $4+2i$ (ii) $4+2i$ (iii) 1 (iv) 4 (v) $1+2i$ (vi) 5

(vii) $0.12-0.16i$ (viii) $-1+2i$ (ix) $-1+2i$

(b) We have $z \times \overline{z} = |z|^2$ (see Exercise 3.2(b)), and so

$$w/z = w\overline{z}/z\overline{z} = w\overline{z}/|z|^2.$$

Mathcad should therefore give the same values for w/z and for $w\overline{z}/|z|^2$, for any complex numbers w and z (where $z \neq 0$).

Mathcad notes

Remember that Mathcad notes are *optional*.

- ◇ $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are built-in Mathcad functions. As well as typing them directly, you can also enter them by selecting **Function...** from the **Insert** menu. Choose 'Complex Numbers' from the 'Function Category' box, then either 'Re' or 'Im' from the 'Function Name' box, before clicking on 'Insert'.
- ◇ The double-quote " (obtained by typing `[Shift]2`) performs several roles in Mathcad. When entered *after* an expression (selected within the blue editing lines) it applies the complex conjugate operator to that expression. However, when " is entered in an empty space in the worksheet (at the red cross cursor) it creates a text region, and when entered in an empty placeholder in an expression it creates a text string variable.

We look next at Argand diagrams, and at translating between the polar and Cartesian forms of a complex number. Mathcad file 221D1-01 provides templates for translating between the two forms. It also shows Mathcad plots illustrating each of these forms.

Activity 6.4 Polar and Cartesian forms

You will not be asked to create such Mathcad plots. (For your interest only, details of how to do so are provided on page 4 of the worksheet.)

Open Mathcad file **221D1-01 Cartesian and polar forms**, and turn to page 2 of the worksheet. Here a and b are the real and imaginary parts of the complex number z, whose Cartesian form is $z = a + bi$. The corresponding polar form is $\langle r, \theta \rangle$, where $r = |z|$ and θ is defined by the Mathcad expression $\arg(z)$, of which more below. In the polar form, θ is by default in radians, but θ is also shown in the worksheet in degrees (which is usually easier to visualise). The page also shows Argand diagram plots illustrating each of the Cartesian and polar forms.

(a) Working by hand, express each of the complex numbers (i)–(iv) below in polar form.

(i) $2 - 2i$ (ii) -5 (iii) $3i$ (iv) $-2 - 2i$

(b) Check that Mathcad gives in each case the result for the polar form that you calculated in part (a). In what range do the values that Mathcad gives for $\arg(z)$ lie?

(c) Use the worksheet to obtain the polar form of $3 - 8i$.

(d) Given a complex number $\langle r, \theta \rangle$ in polar form, whose Cartesian form is $a + bi$, what are its real part a and imaginary part b?

(e) Turn to page 3 of the worksheet, which gives a template for converting from polar to Cartesian form. Check that the equations used here are as you would expect. Use this template to express each of the following in Cartesian form.

(i) $\langle 3, \pi \rangle$ (ii) $\langle 2, 2.1 \rangle$

Comment

(a) We obtain the following polar forms.

(i) $\langle 2\sqrt{2}, -\frac{1}{4}\pi \rangle$ (ii) $\langle 5, \pi \rangle$ (iii) $\langle 3, \frac{1}{2}\pi \rangle$ (iv) $\langle 2\sqrt{2}, -\frac{3}{4}\pi \rangle$

(b) Above the Argand diagram plots, Mathcad gives the same values that were found in part (a), but expressed as decimals (to 3 decimal places), as follows.

(i) $\langle 2.828, -0.785\rangle$ (ii) $\langle 5, 3.142\rangle$ (iii) $\langle 3, 1.571\rangle$

(iv) $\langle 2.828, -2.356\rangle$

Mathcad also gives exact answers, obtained by using symbolic evaluation ($\rightarrow$), further down page 2 of the worksheet.

The values that Mathcad gives for $\arg(z)$ lie in the range $(-\pi, \pi]$. Thus the Mathcad function arg gives the principal value of the argument, as defined in Subsection 3.2. (However, when expressed in degrees, these values differ by 360° from those shown on the lower half of the Argand diagram polar plot.)

Mathcad numbers the angular grid lines in a polar plot from 0° to 360°.

(c) You should obtain the polar form $\langle 8.544, -1.212\rangle$ (to 3 decimal places).

(d) We have $a = r\cos\theta$ and $b = r\sin\theta$.

(e) We obtain the following.

(i) -3 (as we would expect)

(ii) $-1.010 + 1.726i$ (to 3 decimal places)

Now close Mathcad file 221D1-01.

6.2 Finding roots with Mathcad

You have met various ways of using Mathcad to solve equations. Solve blocks and the Newton–Raphson method each employ a numerical algorithm, and each will find only one solution at a time. To find all the roots of a polynomial, both real and complex, a different approach is preferable.

Solve blocks were used in Mathcad file 221B1-03 for Chapter B1. The Newton–Raphson method was applied in Mathcad file 221C1-03 for Chapter C1.

Mathcad will give the two roots of a quadratic polynomial, using the symbolic keyword 'solve'. This works whether the roots are real or complex. The roots are obtained using the formula for the solutions of a quadratic equation, to give exact rather than approximate solutions. If you enter the quadratic with its coefficients given as integers or rationals, then Mathcad gives the roots of the quadratic exactly, using square roots if necessary. If you enter the coefficients as decimals, then Mathcad returns the roots as decimals. Figure 6.1 shows an example. Mathcad initially gives the decimal solutions to 20 places, but if you click on the answer and enter **=**, it displays the solutions to the number of decimal places specified in 'Number of decimal places', on the 'Number Format' tab under **Result...** on the **Format** menu.

$$2x^2 + \frac{5}{2}x + \frac{7}{4} \text{ solve } \rightarrow \begin{pmatrix} -\frac{5}{8} + \frac{\sqrt{31}\,i}{8} \\ -\frac{5}{8} - \frac{\sqrt{31}\,i}{8} \end{pmatrix}$$

$$2x^2 + 2.5x + 1.75 \text{ solve } \rightarrow \begin{pmatrix} -0.625 - 0.69597054535375274026i \\ -0.625 + 0.69597054535375274026i \end{pmatrix} = \begin{pmatrix} -0.625 - 0.696i \\ -0.625 + 0.696i \end{pmatrix}$$

Figure 6.1 Roots obtained using the symbolic keyword 'solve'

This distinction, between exact and decimal expressions, may not seem very important, but it becomes more significant for a cubic polynomial. There is a formula giving the roots of a general cubic (this was touched on in Section 1), but it is much more complicated than the formula for the roots of a quadratic. Mathcad uses that formula if you use the symbolic keyword 'solve' to find the roots of a cubic.

Activity 6.5 Solving a quadratic and a cubic

Create a new (Normal) worksheet for this activity.

(a) Use the symbolic keyword 'solve' to find the roots of the quadratic

$$2x^2 + \tfrac{5}{2}x + \tfrac{7}{4},$$

(i) entering the coefficients as rationals;

(ii) entering the coefficients as decimals.

With the vertical blue editing line anywhere within or at the end of the expression, click on the 'solve' button on the 'Symbolic' toolbar, then either click elsewhere on the worksheet or press `[Enter]`. Alternatively, just type `[Ctrl][Shift].solve`.

(b) Use the symbolic keyword 'solve' to find the roots of the cubic

$$x^3 - x + 2,$$

(i) entering the coefficients as rationals;

(ii) entering one coefficient as a decimal (for example, enter 2 as 2.0).

Comment

(a) You should obtain the results shown in Figure 6.1.

(b) In (i), you will obtain a column of three large expressions that are not easy to read. In (ii), you will obtain decimal expressions with 20 digits. In either case, after clicking on the answer and entering `=`, you should obtain the roots as -1.521 and $0.761 \pm 0.858i$ (to 3 decimal places).

The situation for a quartic polynomial is similar to that for a cubic. There is a formula for the roots, and Mathcad employs this when the symbolic keyword 'solve' is used. However, Mathcad may decide not to show the outcome and to turn the input expression red.

Clicking on the red expression in this case reveals the following message: 'The symbolic result returned is too large to display, but it can be used in subsequent calculations if assigned to a function or variable.'

The formula (with 'solve') is usually an effective way of obtaining all four roots if any coefficient is entered as a decimal.

For polynomials of order five or higher there is no general formula for the roots. In this case, Mathcad may respond in a variety of ways. It is quite good at recognising special situations where it is possible to give all the roots exactly, but in general this is not possible. Sometimes it gives answers with 20 decimal digits, even though none of the coefficients is in decimal form.

Mathcad also has an alternative facility, called 'polyroots', which calculates all the roots of a polynomial, using an iterative procedure. Unlike the symbolic keyword 'solve', polyroots will not give exact answers, but then the exact expressions are often so large as to be unmanageable.

In most cases, polyroots is the most efficient way of finding all the roots of a polynomial. To use polyroots, you need first to enter the coefficients of the polynomial into Mathcad, as a vector. To do this, the coefficients can be entered either as a matrix with one column, or as subscripted variables. For example, to find all the roots of

$$x^5 + 3x^4 - 2x^3 - x^2 + 2x + 1,$$

you could use either of the approaches illustrated in Figure 6.2.

One advantage of polyroots is that it avoids the need to enter all the powers of x involved into the worksheet.

Notice that a_0 (at the top of the matrix) or b_0 gives the constant coefficient, which is 1 in this example; a_1 (second down in the matrix) or b_1 gives the coefficient of x (here 2), and so on. Entering `=`, after clicking on either of the expressions polyroots(a) or polyroots(b), gives the roots.

$$a := \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \\ 3 \\ 1 \end{pmatrix} \qquad b_0 := 1 \quad b_1 := 2 \quad b_2 := -1 \quad b_3 := -2 \quad b_4 := 3 \quad b_5 := 1$$

polyroots(a) polyroots(b)

Figure 6.2 Mathcad screens to find all the roots of $x^5 + 3x^4 - 2x^3 - x^2 + 2x + 1$

To define a matrix or subscripted variable, you can click on the appropriate button on the 'Matrix' toolbar, or type `[Ctrl]m` or `[` (left square bracket), respectively.

Activity 6.6 Finding the roots of a quintic

Create a new (Normal) worksheet for this activity (or continue with your worksheet for Activity 6.5).

(a) Use polyroots to find all the roots of the quintic polynomial

$$x^5 + 3x^4 - 2x^3 - x^2 + 2x + 1.$$

(b) What is the response from Mathcad if you enter the polynomial in part (a) and then apply the symbolic keyword 'solve',

(i) seeking exact solutions;

(ii) seeking approximate solutions, by putting a decimal point into one of the coefficients?

Comment

(a) Figure 6.2 shows the two approaches that can be used here. We obtain the following five roots:

$$-3.454; \quad -0.541 \pm 0.161i; \quad 0.768 \pm 0.566i.$$

(b) (i) If all the coefficients are entered as integers, then Mathcad's response is to give the roots correct to 20 decimal places.

(ii) If one or more coefficients is entered as a decimal, then Mathcad gives identical output to that described in (i).

Activity 6.7 Finding roots

In parts (a) and (b) below we give two results obtained by hand in Section 4 of the main text. Use Mathcad to confirm each of these results.

(a) In Activity 4.4, you found that 8 has the three cube roots

$$2; \quad -1 \pm i\sqrt{3}.$$

(b) In Activity 4.7, you found that $4i$ has the four fourth roots

$$1.307 + 0.541i; \quad -0.541 + 1.307i; \quad -1.307 - 0.541i; \quad 0.541 - 1.307i.$$

Comment

(a) The cube roots of 8 are the roots of the polynomial $x^3 - 8$. The roots have been given exactly, and to obtain exact roots from Mathcad, we should use the symbolic keyword 'solve'. Since we have a cubic polynomial here, this should give all the roots, and it does. (Remember to enter the coefficients as integers.)

(b) The fourth roots of $4i$ are the roots of $x^4 - 4i$. Use of polyroots gives the roots in the form above. (Enter the coefficients $a_0 = -4i$, $a_1 = a_2 = a_3 = 0$ and $a_4 = 1$.) The symbolic keyword 'solve' displays the solutions in an alternative format, which gives the same values to 3 decimal places after entering **=**. If either coefficient is entered as a decimal, then 'solve' produces a column of four zeros.

If you use the same worksheet here as for Activity 6.6, and enter the coefficients using separate subscripted variables, then you may inherit the coefficient $a_5 = 1$ from that work, and polyroots(a) may not give the answer you require! To avoid this problem, use a different name for the subscripted variables in this activity – see the Mathcad notes for an explanation.

Mathcad notes

Care is needed in Mathcad when redefining subscripted variables. Suppose that you define six variables, $a_0, a_1, \ldots, a_5$, but then later in the same worksheet you redefine only five of them, $a_0, a_1, \ldots, a_4$. The sixth value a_5 is still defined, as it is 'inherited' from the first definition. (Note that while this problem may occur when defining separate subscripted variables, it does not occur if the new definitions are made by defining a matrix a with one column.)

You saw how to calculate the nth roots of unity in Section 4. There, we looked at some particular values of n. We can use the same approach to find a general formula for roots of unity. The nth roots of unity satisfy $z^n = 1$. If we express both z and 1 in polar form, as $z = \langle r, \theta \rangle$ and $1 = \langle 1, 0 \rangle$, then this equation gives

$$\langle 1, 0 \rangle = \langle r, \theta \rangle^n = \langle r^n, n\theta \rangle.$$

For this to hold, we need $r = 1$, and $n\theta$ to be a multiple of 2π. Thus, in polar form, the nth roots of unity are $\langle 1, 2k\pi/n \rangle$, where k takes the values $0, 1, \ldots, n-1$. In Cartesian form, they are

$$\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), \quad \text{for } k = 0, 1, \ldots, n-1.$$

The next activity features a Mathcad file to find nth roots of unity.

Activity 6.8 Roots of unity

Open Mathcad file **221D1-02 Roots of unity**, and turn to page 2 of the worksheet. This page is set up to calculate the nth roots of unity (for a given value of n), both using the formula above and using polyroots, to enable you to compare the results. The coefficients $a_0, a_1, \ldots, a_{n-1}$ are defined by means of the Mathcad function if(*condition*, p, q).

(a) Use this page to find the ninth roots of unity.

(b) Examine the Argand diagram plot of the nth roots of unity, on page 3 of the worksheet, for n equal to each of 5, 10, 25 and 50, and note any patterns that you observe. (You will need to set the value of n on page 2, in each case.)

Comment

(a) You need to set n equal to 9. The formula and polyroots give the same roots (though in different orders). The ninth roots of unity are

$$1,\ 0.766 \pm 0.643i,\ 0.174 \pm 0.985i,\ -0.5 \pm 0.866i,\ -0.940 \pm 0.342i.$$

(b) The roots of unity are always evenly spaced around the unit circle, as mentioned in Section 4. For larger values of n, the plot of corresponding points looks increasingly like a circle. (You may have noted other features.)

Mathcad notes

The tables of roots on page 2 of the worksheet have been formatted to display them in the same style. By default, entering 'polyroots(a) =' displays nine values or less within round brackets (in matrix form). To force Mathcad to display a table here, for any number of values, we have used the **Format** menu, **Result...** , and on the 'Display Options' tab, changed the 'Matrix display style' from 'Automatic' to 'Table'. Further changes have been made by clicking on the 'polyroots' table with the *right* mouse button, and choosing first 'Properties...', then 'Alignment' from the resulting mini-menu. In the table properties, column/row labels are not shown, and the alignment (of the expression 'polyroots(a) =' to the table values) is set to 'Top'. This alignment has also been set for the other two tables on the page, 'k =' and 'z_k ='.

Now close Mathcad file 221D1-02.

6.3 Complex exponentials and geometry

Complex exponentials are entered into Mathcad in just the same way as real exponentials, for example, by typing `e^z` (or `exp(z)`).

This subsection will not be assessed.

Activity 6.9 Complex exponentials (Optional)

(a) In a new worksheet, set $z = 2 + 3i$, then evaluate e^z.

(b) With $z = 2 + 3i$, what are $|e^z|$ and $\arg(e^z)$? Use Mathcad to verify your answers.

Comment

(a) We obtain $e^{2+3i} = -7.315 + 1.043i$. (You found this result by hand in Activity 5.2.)

(b) We have $e^z = e^{2+3i} = e^2(\cos 3 + i \sin 3)$. So

$$|e^z| = e^2, \quad \arg(e^z) = 3.$$

Mathcad confirms these results. We obtain $|e^z| = 7.389$, and this equals e^2 to 3 decimal places. Symbolic evaluation ($\rightarrow$) of $|e^z|$ gives e^2 directly.

We now use Mathcad to investigate sequences generated by the recurrence system

$$c_0 = 1, \quad c_{n+1} = kc_n \quad (n = 0, 1, 2, \ldots), \tag{6.1}$$

where k is a fixed complex number. (We considered such sequences in Subsection 5.2 of the main text.)

Activity 6.10 Iterations with $|k|=1$ (Optional)

Open Mathcad file **221D1-03 Complex exponentials** and turn to page 2 of the worksheet. Here we consider the recurrence system (6.1) with k of the form $e^{i\theta}$, so that $|k| = 1$.

(a) Consider first the sequence generated by the recurrence system (6.1) with $k = e^{i\pi/6}$.

(i) This sequence will eventually repeat itself. How do we know this?

(ii) How many iterations are needed before the sequence starts to repeat? Set N on page 2 of the worksheet to the smallest value that will plot all points of the sequence. Check that larger values of N plot nothing new.

(b) For what values of θ in $k = e^{i\theta}$ does the sequence generated by the recurrence system (6.1) eventually repeat?

(c) Decide on a value of θ for which the plot should look close to a circle. Choose a value of θ for which the sequence eventually repeats, and choose N large enough to show all the points of the sequence. Enter these values on page 2 of the worksheet, and check that they have the effect that you expected.

(d) Decide on a value of θ for which the sequence does not eventually repeat. Examine the plot on page 2 for your chosen value of θ.

Comment

(a) (i) You saw in Activity 5.5 of the main text that the sequence generated by the recurrence system (6.1) eventually repeats if and only if k is a root of unity. Now

$$(e^{i\pi/6})^{12} = e^{(i\pi/6)\times 12} = e^{2i\pi} = \cos(2\pi) + i\sin(2\pi) = 1,$$

so $e^{i\pi/6}$ is a twelfth root of unity.

(ii) We have $c_{12} = k^{12} = 1 = c_0$, and after this the terms of the sequence repeat. So with $N = 11$ on page 2 of the worksheet, we see all points of the sequence plotted. With $N = 12$, the line joining c_{11} and c_{12} completes a regular 12-sided polygon, and the Mathcad plot looks the same for all higher values of N.

(b) The sequence eventually repeats if k is a root of unity. Now $k = e^{i\theta}$ is a root of unity if $k^m = 1$ for some integer m; that is, if we can find an integer m such that

$$(e^{i\theta})^m = 1, \quad \text{or} \quad e^{mi\theta} = 1.$$

This is the case if $m\theta$ is equal to 2π, or to an integer multiple of 2π. Thus the sequence generated by the recurrence system (6.1) with $k = e^{i\theta}$ eventually repeats if

$$\theta = 2p\pi/m,$$

where p and m ($\neq 0$) are integers.

(c) The plot will eventually repeat if we choose $k = e^{i\theta}$ with $\theta = 2p\pi/m$. It will look close to a circle if we choose m reasonably large, say $m = 100$, and for simplicity we may as well take $p = 1$. So take k equal to, say, $e^{i\pi/50}$ and $N = 99$. (There are many other possible choices of k and N.)

(d) Any value of k that is not a root of unity will produce a sequence that does not repeat. For example, for $k = e^{i\theta}$ with $\theta = 1$, and $N = 7$, we obtain the plot shown in Figure 6.3(a). You can see that k^7 is not equal to 1, and the plot does not return to its starting point. Nor does it return to 1 later. For example, the plot with $N = 13$ is shown in Figure 6.3(b). With larger values of N, as illustrated in Figure 6.3(c) with $N = 100$, it is much harder to make out what is happening, but the iteration never returns exactly to its starting point of $c_0 = 1$.

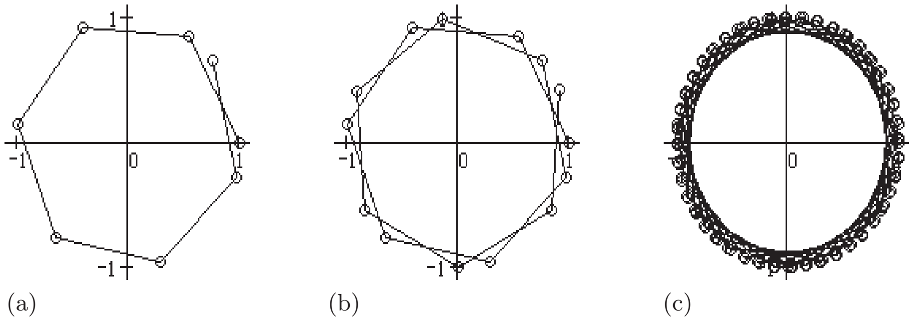


Figure 6.3 Plots of the sequence generated by recurrence system (6.1) with $k = e^i$, showing the sequence up to k^N with N equal to: (a) 7 (b) 13 (c) 100

Mathcad notes

To enter the complex exponential $re^{i\theta}$ in Mathcad, you can use the buttons on the 'Calculator' and 'Greek' toolbars, or type `r*e∧1i*q[Ctrl]g`. (You must type `1i` and include the multiplication `*` between the i and the θ.)

Activity 6.11 The twentieth roots of unity (Optional)

You should still be working with Mathcad file 221D1-03, on page 2 of the worksheet.

(a) For what values of θ is $e^{i\theta}$ a twentieth root of unity?

(b) Examine the sequences generated by the recurrence system (6.1) with $k = e^{ip\pi/10}$, where p is equal to each of

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 17,\ 18,\ 19.$$

Note any points that you observe about the plots you obtain.

Comment

(a) We have $(e^{i\theta})^{20} = 1$ if $e^{20i\theta} = 1$. This is the case if 20θ is an integer multiple of 2π; that is, if $\theta = p\pi/10$, where p is an integer. (We obtain the complete set of twentieth roots of unity if p takes the values $0, 1, 2, \ldots, 19$; other values of p just repeat one of these.)

(b) Plots for $p = 1$, 2 and 3 are shown in Figure 6.4 (for $N = 20$ iterations). For $p = 1$ the twentieth roots of unity are uniformly spaced around a unit circle, as we would expect. For $p = 2$, we obtain fewer points, 10 rather than 20. This is because $2\pi/10 = \pi/5$, and $(e^{\pi/5})^{10} = e^{2\pi} = 1$. Hence $e^{2\pi/10}$ is a tenth root of unity (as well as being a twentieth root). So the plot in this case repeats itself after 10 iterations, and not all the twentieth roots of unity are visited. With $p = 3$, all the twentieth roots of unity are again visited by the plot. The order in which they are visited is different, though. With $p = 1$, the roots are visited in order as we go round the circle, that is, with m equal to $0, 1, 2, \ldots, 19$ in $e^{im\pi/20}$. With $p = 3$, the roots are visited in the order

$$0,\ 3,\ 6,\ 9,\ 12,\ 15,\ 18,\ 1,\ 4,\ 7,\ 10,\ 13,\ 16,\ 19,\ 2,\ 5,\ 8,\ 11,\ 14,\ 17.$$

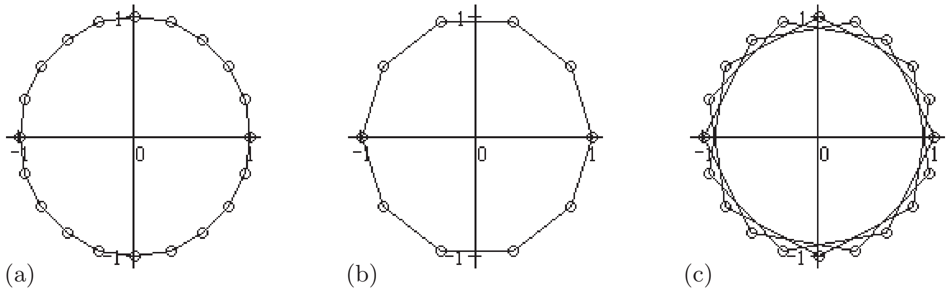


Figure 6.4 Plots of sequences generated by recurrence system (6.1) with $k = e^{ip\pi/10}$, for p equal to: (a) 1 (b) 2 (c) 3

With $p = 4$, we see only five roots. Here $4\pi/10 = 2\pi/5$, and $e^{2i\pi/5}$ is a fifth root of unity.

With $p = 5$, we see only four roots, since $5\pi/10 = \pi/2$, and $e^{i\pi/2}$ is a fourth root of unity.

With $p = 6$, we see 10 roots, since $6\pi/10 = 3\pi/5$, and $e^{3i\pi/5}$ is a tenth root of unity. Here the roots are visited in the order (labelling round the circle): 0, 3, 6, 9, 2, 5, 8, 1, 4, 7.

With $p = 7$, all 20 roots are visited. Notice that $7\pi/10$ does not simplify at all, since 7 and 10 have no common factors. So here $e^{7i\pi/10}$ is a twentieth root of unity, but not an nth root for any smaller value of n.

With $p = 17$, we see the same plot as for $p = 3$. Although you cannot see it from the plot, in this case the roots are actually visited in the reverse of the order for $p = 3$, that is, $0, 17, 14, 11, \ldots, 3$.

With $p = 18$, we see the same plot as for $p = 2$; and for $p = 19$, we obtain the same plot as for $p = 1$.

If $|k|$ is not equal to 1, then the recurrence system (6.1) produces a spiral, as you saw in Subsection 5.2.

Activity 6.12 Spirals (Optional)

Consider now sequences generated by the recurrence system (6.1) for a general complex number k, with polar form $\langle r, \theta \rangle$ where $r \neq 1$.

You should still be working with Mathcad file 221D1-03, on page 2 of the worksheet.

(a) Examine the plot of the sequence with the following values of $k = \langle r, \theta \rangle$.

(i) $\langle 1.1, \pi/6 \rangle$ (ii) $\langle \sqrt{2}, \pi/2 \rangle$ (iii) $\langle 1/\sqrt{2}, \pi/2 \rangle$

(iv) $\langle \sqrt{2}, -\pi/2 \rangle$

In each case try two or three values for N. You will need to adjust the value of the graph scale variable s appropriately for each value of k.

(b) Choose your own values of r and θ in $k = \langle r, \theta \rangle$. How does the spiral vary?

Comment

(a) You saw plots of the spirals in these cases in Figures 5.2 and 5.3 of the main text.

(b) With $r > 1$, the plot spirals outwards, and the larger r is, the more rapidly the spiral moves out. If $r < 1$, the spiral moves inwards. If $0 < \theta < \pi$ the spiral goes round anticlockwise, while if $-\pi < \theta < 0$ it goes round clockwise.

Finally, we look at plots of the complex-valued function $f(t) = r(t)e^{it}$. With $r(t) = a^t$, this plot gives a continuous spiral.

Activity 6.13 Continuous spirals (Optional)

Turn to page 3 of the worksheet, which shows a plot of the complex-valued function $f(t) = r(t)e^{it}$ $(t \geq 0)$.

You should still be working with Mathcad file 221D1-03.

(a) Examine the plot with $r(t) = a^t$, where a is equal to each of the following.

(i) 1.1 (ii) 1.2 (iii) 0.9 (iv) 0.8 (v) 1

How does the plot vary? (You will need to adjust the value of the graph scale variable s in order to see the plots for (iii) and (iv) well.)

(b) Examine the plot with $r(t)$ equal to each of the following.

(i) t (ii) $1 + 0.2\cos t$ (iii) $\ln t$ $(t \geq 1)$

How does the plot vary? (Again, adjust s appropriately.)

Comment

(a) With $a > 1$, the plot is an increasing spiral. The rate of increase is faster if a is larger. With $a < 1$, the plot is a decreasing spiral. The rate of decrease is faster if a is smaller. With $a = 1$, the plot is a circle.

(b) (i) With $r(t) = t$, we again see an increasing spiral.

(ii) In this case the plot is a closed curve.

(iii) Here you need to change the range for t, setting $T1 := 1$, because the domain given is $t \geq 1$. We see an increasing spiral, but the rate of increase becomes slower and slower. (This is particularly clear if you modify the range for t, to give a larger upper limit, such as $T2 := 40$.)

Now close Mathcad file 221D1-03.

Chapter D2, Section 5
Number theory and Mathcad

In this section, you will have an opportunity to develop your understanding of the concepts in the main text, to check some of your earlier calculations, and also to try out the methods with much larger numbers than your calculator can handle.

Mathcad can perform some arithmetic operations with very large integers by using symbolic evaluation, but we do not describe these capabilities until the end of the section.

5.1 The Division Algorithm

See Theorem 1.1 of the main text.

The Division Algorithm asserts that if a and n are integers, with n positive, then there are unique integers q and r such that

$$a = qn + r, \quad \text{with } 0 \le r < n. \tag{5.1}$$

The number q is called the *quotient* and r is called the *remainder* of a, on division by n.

Equation (5.1) leads to formulas for q and r:

As you saw in Subsection 1.1 of the main text, floor(x), also denoted by $[x]$, is the greatest integer less than or equal to x.

$$q = \operatorname{floor}\left(\frac{a}{n}\right) \quad \text{and} \quad r = a - qn.$$

The first activity in this section asks you to try out these two formulas in Mathcad.

Activity 5.1 Quotients and remainders

(a) By hand, find the quotient and remainder of

(i) 32 on division by 6; (ii) -32 on division by 6.

(b) Create a new (Normal) worksheet, and enter the following expressions.

$$a := 32 \qquad n := 6$$

$$q := \operatorname{floor}\left(\frac{a}{n}\right) \qquad r := a - q\,n$$

Make sure that you place the formula for r after that for q in the worksheet.

Then enter `q=` and `r=` to evaluate these quantities. Change the value of a to -32, and check that Mathcad gives the same answers that you obtained in part (a).

Comment

(a) (i) $q = 5$, $r = 2$ (ii) $q = -6$, $r = 4$

(b) Mathcad gives the same values.

It is a good idea to enter some text to structure the worksheet you are creating in this and subsequent activities. For example, a title and some headings and comments might be useful.

Activity 5.1 shows that Mathcad can easily be used to find quotients and remainders. To investigate the behaviour of these quotients and remainders, it is convenient to introduce a pair of functions defined as follows.

$$\mathrm{quot}(a,n) := \mathrm{floor}\left(\frac{a}{n}\right) \qquad\qquad \mathrm{rem}(a,n) := a - n\,\mathrm{quot}(a,n)$$

Activity 5.2 Varying a and keeping n fixed

(a) Enter the functions above in your Mathcad worksheet, and check that $\mathrm{quot}(-32,6) = -6$ and $\mathrm{rem}(-32,6) = 4$.

(b) Describe how $\mathrm{quot}(a,n)$ and $\mathrm{rem}(a,n)$ behave when n is kept fixed and a is allowed to increase. It may help to consider an example, such as $n := 3$ and $a := -10, -9 \,..\, 10$, and to create tables or graphs for $\mathrm{quot}(a,n)$ and $\mathrm{rem}(a,n)$. (You may need to format graphs appropriately to show the behaviour clearly.)

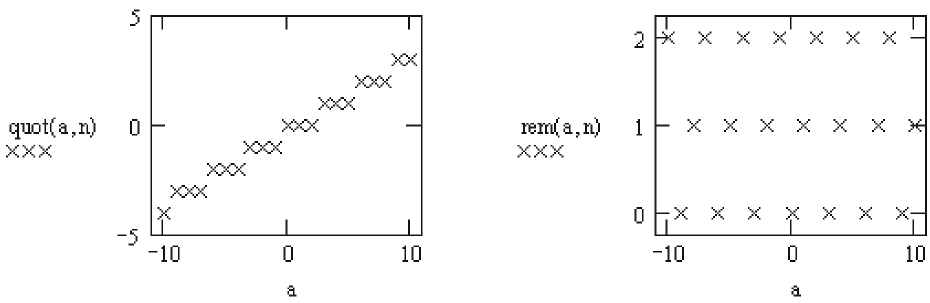
Comment

(b) As a increases

◇ $\mathrm{quot}(a,n)$ increases by 1 whenever a reaches a multiple of n, but is otherwise equal to its previous value;

◇ $\mathrm{rem}(a,n)$ cycles repeatedly through the numbers $0, 1, 2, \ldots, n-1$.

These statements are illustrated by the graphs in Figure 5.1.



(a) (b)

Figure 5.1 Graphs of the functions (a) quot (b) rem

You may have wondered whether Mathcad has its own built-in functions for calculating quotients and remainders. In fact Mathcad has a function called mod, which can be used for finding remainders. This function has two arguments: it can be obtained by typing `mod(a,n)`.

Activity 5.3 The function mod

(a) Enter the following expressions in your worksheet, and then evaluate them by entering `=`.

(i) $\mathrm{mod}(32,6)$ (ii) $\mathrm{mod}(-32,6)$

(b) Vary the numbers used as arguments in part (a). On the basis of the observed values, attempt to define the function mod.

(c) What happens when you try to evaluate $\mathrm{mod}(a^k, n)$, where $a = 3$, $k = 1000$ and $n = 13$?

Comment

(a) (i) $\text{mod}(32, 6) = 2$ (ii) $\text{mod}(-32, 6) = -2$

(b) Several illuminating examples are:

$$\text{mod}(0, 6) = 0, \quad \text{mod}(32, -6) = 2, \quad \text{mod}(-32, -6) = -2.$$

It seems from these examples that Mathcad evaluates $\text{mod}(a, n)$, where a and n are integers with $n \neq 0$, by first finding the remainder when $|a|$ is divided by $|n|$, and then giving this remainder the same sign as a.

Mathcad will also return a value for $\text{mod}(a, n)$ when a and n are not integers, but that does not concern us here.

(c) An error message is displayed: 'Found a number with a magnitude greater than 10^307 while trying to evaluate this expression.'. Finding such remainders, when a^k is large, is addressed in Activities 5.4 and 5.6.

Save the worksheet that you have created, if you wish. Then close the file.

An alternative definition of the Mathcad function mod is as follows:

This definition applies also to symbolic evaluation (with $\rightarrow$) of $\text{mod}(a, n)$.

$\text{mod}(a, n)$ is the unique integer r satisfying

- ◇ $|r| < |n|$;
- ◇ $a = qn + r$, for some integer q;
- ◇ r has the same sign as a.

This makes it clear that $\text{mod}(a, n)$ is a remainder of a on division by n, but for $a < 0$ it is *not* the remainder which is useful in number theory. However, we can use $\text{mod}(a, n)$ to find such remainders when a and n are both positive.

5.2 Remainders of powers

In Section 1 of the main text two methods, here called algorithms, were described for finding the remainder of a^k on division by n, where a, k and n are positive integers. Both these algorithms – repeated multiplication and repeated squaring – are implemented in Mathcad file 221D2-01.

Repeated multiplication

To find the remainder of a^k on division by n, we calculate successively the remainders of $a^0, a^1, a^2, a^3, \ldots$ on division by n, using a recurrence relation which expresses each remainder in terms of the previous one.

The sequence of remainders always shows a repeating pattern.

If we denote by r_i the remainder of a^i on division by n, then

$$r_i \equiv a^i \pmod{n},$$

so

$$r_i \equiv a \times a^{i-1} \equiv ar_{i-1} \pmod{n}.$$

Also, $r_0 \equiv a^0 \equiv 1 \pmod{n}$. Thus there is a simple recurrence system for calculating the remainders $r_0, r_1, r_2, \ldots, r_k$.

Activity 5.4 Repeated Multiplication Algorithm

Open Mathcad file **221D2-01 Remainders of powers**, and turn to page 2 of the worksheet. This implements the Repeated Multiplication Algorithm by means of a Mathcad program. The page is set up with $a = 3$, $k = 1000$ and $n = 13$, and shows that the remainder of $a^k = 3^{1000}$ on division by 13 is $r_k = 3$. The table of remainders illustrates the repeating pattern of values within the sequence $r_0, r_1, r_2, \ldots, r_k$.

(a) Use the worksheet to find the remainder on division of 5^{2000} by 21.

(b) Experiment with various choices of a and n (keeping k fixed at 2000), to find repeating sequences of remainders with the following forms.

(i) $1, r, r, r, \ldots$ (ii) $1, r, s, r, s, \ldots$ (iii) $1, r, s, t, r, s, t, \ldots$

Comment

(a) With $a = 5$, $k = 2000$ and $n = 21$, the required remainder is $r_{2000} = 4$. Note that the table shows the sequence

$$1,\ 5,\ 4,\ 20,\ 16,\ 17,\ 1,\ 5,\ 4,\ 20,\ 16,\ 17, \ldots,$$

with a pattern which repeats after 6 terms. If you scroll across the table to $i = 2000$, you will see that the remainder $r_{2000} = 4$ appears there as well.

A 'by hand' check is as follows. Since $5^6 \equiv 1 \pmod{21}$, we obtain

$$5^{2000} \equiv 5^{6\times 333+2} \equiv (5^6)^{333} \times 5^2 \equiv 5^2 \equiv 25 \equiv 4 \pmod{21}.$$

(b) (i) $a = 4$, $n = 12$ gives remainders $1, 4, 4, \ldots$

(ii) $a = 3$, $n = 12$ gives remainders $1, 3, 9, 3, 9, \ldots$

(iii) $a = 2$, $n = 14$ gives remainders $1, 2, 4, 8, 2, 4, 8, \ldots$

These answers are not unique.

The next activity revisits a result from Section 3 of the main text.

Activity 5.5 Experiment with primes

(a) Use the worksheet again to experiment with various prime numbers for n and positive integers a, where a is not a multiple of n. (Put $k = 1000$.) Notice from the table that 1 usually appears in the sequence of remainders, and try to explain this.

You should still be working with Mathcad file 221D2-01, on page 2 of the worksheet.

(b) Try $n = 341$ (which is not prime) and $a = 2$, and check that in this case also, 1 appears in the sequence of remainders.

Comment

(a) If n is a prime number p and a is not a multiple of p, then we expect 1 to appear in the sequence of remainders, by Fermat's Little Theorem (Theorem 3.2 in the main text):

> Let p be a prime number, and let a be a positive integer which is not a multiple of p. Then $a^{p-1} \equiv 1 \pmod{p}$.

(b) It can be seen from the table that $2^{10} \equiv 1 \pmod{341}$, which can also be checked by hand. This shows that 1 may appear in the sequence of remainders, even when n is *not* a prime number.

$2^{10} = 1024 = 3 \times 341 + 1$

Repeated squaring

A more efficient algorithm for finding the remainder of a^k on division by n involves the steps set out below. (Some of the details are illustrated in the margin for the case $a^k = 14^{27}$, $n = 55$, discussed in Subsection 1.3 of the main text.)

$$27 = 1+2+8+16 = 1+1\times 2+0\times 2^2 + 1\times 2^3 + 1\times 2^4$$

◇ Represent k as a sum of powers of 2:

$$k = c_0 + c_1 \times 2^1 + c_2 \times 2^2 + \cdots + c_m \times 2^m, \qquad (5.2)$$

where $c_0, c_1, \ldots, c_m$ are in $\mathbb{Z}_2$ and $c_m = 1$.

For $a = 14$ we have, for example, $s_1 = 31$ and so

$$s_2 \equiv 31^2 \equiv 26 \pmod{55}.$$

◇ Find the remainders $s_0, s_1, s_2, \ldots, s_m$ on division by n of $a^1, a^2, a^4, \ldots, a^{2^m}$, using repeated squaring:

$$s_i \equiv s_{i-1}^2 \pmod{n}.$$

$$14^{27} \equiv 14^1 \times 14^2 \times 14^8 \times 14^{16} \equiv 14^1 \times 31^1 \times 16^1 \times 36^1 \pmod{55}$$

◇ Find the remainder on division of a^k by n:

$$\begin{aligned} a^k &\equiv a^{c_0} \times (a^2)^{c_1} \times (a^4)^{c_2} \times \cdots \times (a^{2^m})^{c_m} \\ &\equiv s_0^{c_0} \times s_1^{c_1} \times s_2^{c_2} \times \cdots \times s_m^{c_m} \pmod{n}, \end{aligned}$$

using repeated multiplication modulo n.

This algorithm is implemented on the next page of the Mathcad worksheet. It is a little complicated, particularly the first step which finds the representation of k as a sum of powers of 2, the so-called *binary representation* of k. How the implementation of the algorithm works is explained after the next activity.

Activity 5.6 Repeated Squaring Algorithm

You should still be working with Mathcad file 221D2-01.

The role of the sequence $p_0, p_1, \ldots, p_m$ is described after this activity.

Turn to page 3 of the worksheet. This implements the Repeated Squaring Algorithm by means of a Mathcad program. The page is set up with $a = 14$, $k = 27$ and $n = 55$, and shows that the remainder of $a^k = 14^{27}$ on division by 55 is $p_m = 9$. The table displays intermediate values generated by the algorithm.

Use the worksheet to find each of the following:

(a) the remainder of $2^{10\,000}$ on division by $10\,001$;

(b) the remainder of 2^{561} and of 3^{561} on division by 561.

Comment

(a) Here you need to enter $a = 2$, $k = 10\,000$ and $n = 10\,001$. The algorithm gives the remainder as $p_m = 4674$, so $2^{10\,000} \equiv 4674 \pmod{10\,001}$.

(This result shows that 10 001 is *not* a prime number, because if it were then the remainder would have been 1, by Fermat's Little Theorem. As you saw in Subsection 3.3 of the main text, we have $10\,001 = 73 \times 137$.)

(b) Here the remainders p_m are 2 and 3, respectively, so $2^{561} \equiv 2 \pmod{561}$ and $3^{561} \equiv 3 \pmod{561}$.

$561 = 3 \times 11 \times 17$

(This result suggests that $a^{561} \equiv a \pmod{561}$, for *every* positive integer a, even though 561 is not prime, and this is true. A proof of this fact can be based on the congruences

$$a^2 \equiv 1 \pmod{3}, \quad a^{10} \equiv 1 \pmod{11}, \quad a^{16} \equiv 1 \pmod{17},$$

which hold if a is coprime with 561.)

We now give an explanation of the program for the Repeated Squaring Algorithm, given on page 3 of the worksheet.

The rest of this subsection will not be assessed.

You are not expected to be able to construct such an algorithm, but if you have time then try to see how it works.

The input values a, k and n are declared for use in the program. The method of calculating the numbers $c_0, c_1, \ldots, c_m$, in the binary representation (5.2) of k, differs from that used in Subsection 1.3 (it avoids having to find the largest power of 2 which is less than k). First note that $q_0 = k$ and $c_0 = \text{mod}(k, 2)$.

The numbers $c_1, c_2, \ldots, c_m$ are found by calculating the repeated quotients on division by 2:

$$\begin{aligned}
q_1 &= \text{floor}\left(\tfrac{1}{2}q_0\right) = c_1 + c_2 \times 2^1 + \cdots + c_m 2^{m-1}, &\quad \text{so } c_1 &= \text{mod}(q_1, 2);\\
q_2 &= \text{floor}\left(\tfrac{1}{2}q_1\right) = c_2 + \cdots + c_m 2^{m-2}, &\quad \text{so } c_2 &= \text{mod}(q_2, 2);\\
&\ \vdots & &\ \vdots\\
q_m &= \text{floor}\left(\tfrac{1}{2}q_{m-1}\right) = c_m, &\quad \text{so } c_m &= \text{mod}(q_m, 2).
\end{aligned}$$

The counter i keeps track of how many iterations of this type take place. The final iteration gives $q_i = 1$, and the variable m is defined as the corresponding value of i, so m denotes the number of iterations that have occurred.

These iterations take place within a 'while loop', which is the block of program steps with a solid vertical line to the left and headed by the line 'while $q_i > 1$'. Also within this loop, Mathcad calculates by repeated squaring the remainders on division by n of $a^2, a^4, \ldots, a^{2^m}$, which are denoted respectively by $s_1, s_2, \ldots, s_m$. (The remainder of a on division by n is given, before the while loop starts, by $s_0 = \text{mod}(a, n)$.)

The values $s_0, s_1, \ldots, s_m$ are used to calculate the sequence of products

$$p_0 = s_0^{c_0},\ p_1 = s_0^{c_0} \times s_1^{c_1}, \ldots,\ p_m = s_0^{c_0} \times s_1^{c_1} \times \cdots \times s_m^{c_m},$$

modulo n, again by iteration. The final product p_m is the required remainder.

The program also outputs the values of m and c_i, s_i, p_i $(i = 0, 1, \ldots, m)$, so details of the calculation can be displayed in a table.

Now close Mathcad file 221D2-01.

5.3 Arithmetic in $\mathbb{Z}_n$

Mathcad's mod function makes it easy to obtain addition and multiplication tables for $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ since, for example,

$$a +_n b = \text{mod}(a + b, n), \quad \text{when } a, b \in \mathbb{Z}_n.$$

Activity 5.7 Addition and multiplication in $\mathbb{Z}_n$

Open Mathcad file **221D2-02 Modular arithmetic**, and turn to page 2 of the worksheet.

The multiplication will be displayed as $a \times b$, since '$\times$' is the default (**Tools** menu, **Worksheet Options...**, 'Display') for viewing multiplications in this worksheet.

(a) Remind yourself of the overall pattern in addition tables for $\mathbb{Z}_n$. Try changing n to 9, 10 and 12, say.

(b) Change addition to multiplication in the definition of $p_{a,b}$, and use the table for multiplication in $\mathbb{Z}_{11}$ to find the multiplicative inverse of 5 in $\mathbb{Z}_{11}$.

(c) Use the multiplication table for $\mathbb{Z}_{26}$ to find the multiplicative inverses of 7 and 9 in $\mathbb{Z}_{26}$.

(d) Which rows of the multiplication table for $\mathbb{Z}_{26}$ include 1 and therefore all of $\mathbb{Z}_{26}$? What do you notice about each of these row numbers and the number 26? Relate your observations to Theorem 3.1 of the main text.

Comment

(a) The addition table for $\mathbb{Z}_n$ has the 'constant diagonal' pattern.

(b) The multiplicative inverse of 5 in $\mathbb{Z}_{11}$ is 9.

(c) The multiplicative inverse of 7 in $\mathbb{Z}_{26}$ is 15, and the multiplicative inverse of 9 in $\mathbb{Z}_{26}$ is 3.

(d) The rows of the multiplication table for $\mathbb{Z}_{26}$ which include all of $\mathbb{Z}_{26}$ are: 1, 3, 5, 7, 9, 11, 15, 17, 19, 21, 23, 25. These are the only integers in $\mathbb{Z}_{26}$ which are coprime with 26. These observations verify Theorem 3.1 of the main text for the case $n = 26$.

Finding multiplicative inverses in sets $\mathbb{Z}_n$ is important, for example in Section 4 on cryptography. If n is not too large then multiplicative inverses can be read off from the appropriate multiplication table, but when n is larger a method based on Euclid's Algorithm is available.

Suppose that we wish to find the multiplicative inverse of a in $\mathbb{Z}_n$, where a and n are coprime. Euclid's Algorithm involves

◇ applying the Division Algorithm first to n and a, and then repeatedly to pairs of remainders until the remainder 0 occurs:

$$\left.\begin{aligned} n &= q_1 a + r_1 \\ a &= q_2 r_1 + r_2 \\ r_1 &= q_3 r_2 + r_3 \\ &\vdots \\ r_{m-2} &= q_m r_{m-1} + r_m \\ r_{m-1} &= q_{m+1} r_m \end{aligned}\right\} \tag{5.3}$$

(since a and n are coprime, we must have $r_m = 1$);

◇ eliminating the remainders $r_1, r_2, \ldots, r_{m-1}$ from all but the last of the above equations, to obtain an equation of the form

$$ab = kn + 1, \quad \text{with } b \text{ in } \mathbb{Z}_n.$$

Then b is the required multiplicative inverse of a in $\mathbb{Z}_n$.

The algorithm for multiplicative inverses is the subject of the next activity.

Activity 5.8 Multiplicative Inverse Algorithm

Turn to page 3 of the worksheet. This implements the Multiplicative Inverse Algorithm by means of a Mathcad program. The page is set up with $a = 7$ and $n = 26$, and shows that the multiplicative inverse of 7 in $\mathbb{Z}_{26}$ is $b = 15$. The tables display intermediate values generated by the algorithm.

You should still be working with Mathcad file 221D2-02.

Use the worksheet to find the multiplicative inverse of each of the following.

(a) 8 in $\mathbb{Z}_{19}$ (b) 19 in $\mathbb{Z}_{100}$ (c) 75 in $\mathbb{Z}_{49\,139}$

Comment

(a) Setting a equal to 8 and n equal to 19 gives the multiplicative inverse of 8 in $\mathbb{Z}_{19}$ to be 12.

(b) The multiplicative inverse of 19 in $\mathbb{Z}_{100}$ is 79.

(c) The multiplicative inverse of 75 in $\mathbb{Z}_{49\,139}$ is 10 483.

An alternative method of finding the answer to part (a) is to use the fact that 19 is prime, so

$$8^{18} \equiv 1 \pmod{19},$$

by Fermat's Little Theorem. Hence the multiplicative inverse of 8 in $\mathbb{Z}_{19}$ is congruent to 8^{17} modulo 19, and this can be calculated using, for example, the Repeated Squaring Algorithm. A similar approach can be used in part (c), because 49 139 is prime.

We now explain the program for the Multiplicative Inverse Algorithm, given on page 3 of the worksheet. The quotients and remainders in Euclid's Algorithm are found by putting $r_0 = a$ and then using the iteration

The rest of this subsection will not be assessed.

Again, you are not expected to be able to construct such algorithms.

$$r_1 = \operatorname{mod}(n, a), \qquad q_1 = \operatorname{floor}\left(\frac{n}{a}\right),$$

$$r_i = \operatorname{mod}(r_{i-2}, r_{i-1}), \quad q_i = \operatorname{floor}\left(\frac{r_{i-2}}{r_{i-1}}\right).$$

The iteration continues while $r_i > 1$, after which m is put equal to the final value of i.

We know that $r_m = 1$ (provided that a and n are coprime, as assumed). The other remainders r_i are eliminated from equations (5.3) by multiplying by suitable constants $c_1, c_2, \ldots, c_{m+1}$, and then adding up all $m + 1$ equations. To eliminate the r_j (for $j < m$) we require the constants to satisfy

$$c_{j+2} = q_{j+1}c_{j+1} + c_j.$$

Thus we want

$$c_j = -q_{j+1}c_{j+1} + c_{j+2}, \quad \text{for } j = 1, 2, \ldots, m-1. \tag{5.4}$$

The remaining equation, of terms obtained by summing $m + 1$ equations as above but not eliminated by equations (5.4), is

$$c_1 n + c_2 a = c_1 q_1 a + (c_m + c_{m+1}q_{m+1})r_m.$$

We know that $r_m = 1$, so if we choose $c_m = 1$ and $c_{m+1} = 0$, then this becomes

$$c_1 n + c_2 a = c_1 q_1 a + 1; \quad \text{that is,}$$
$$a(-c_1 q_1 + c_2) = (-c_1)n + 1.$$

Defining $c_0 = -q_1c_1 + c_2$ gives $ac_0 \equiv 1 \pmod{n}$, but it also gives an equation that fits the pattern of equations (5.4) for $j = 0$. Thus we have the recurrence system

$$c_{m+1} = 0, \quad c_m = 1,$$
$$c_j = -q_{j+1}c_{j+1} + c_{j+2}, \quad \text{for } j = m-1, m-2, \ldots, 0.$$

This recurrence system is implemented in the Mathcad program.

Since $ac_0 \equiv 1 \pmod{n}$, the multiplicative inverse b of a in $\mathbb{Z}_n$ is congruent to c_0 modulo n; that is,

$$b = c_0 - n \operatorname{floor}\left(\frac{c_0}{n}\right).$$

The value of b is output by the program. The values of m, r_i, q_i and c_j are also output, so the details of the calculation can be displayed in two tables.

Now close Mathcad file 221D2-02.

5.4 Multiple precision arithmetic

In Section 4 of the main text, you saw how arithmetic with large integers is used in cryptography. In particular, you saw the relevance of large prime numbers to a method of public key cryptography based on Fermat's Little Theorem. Symbolic evaluation in Mathcad can be used to perform arithmetic with large integers, so-called *multiple precision arithmetic*, and also to discover large prime numbers.

Arithmetic

For most numerical calculations, Mathcad maintains 15 digits of precision. However, for symbolic evaluations, basic arithmetic operations can be performed with rational numbers to many hundreds of digits of precision.

Activity 5.9 Arithmetic and mod with the symbolic processor

Care is needed in Mathcad when entering large numbers and reading them in answers. The digits are displayed in a long string – Mathcad does *not* help by grouping them in threes, with gaps between, as is usual in text.

To enter the factorial in part (f), click on the $n!$ button on the 'Calculator' toolbar, or type `!` (given by `[Shift]1`).

Create a new (Normal) worksheet. Enter each of the following expressions, then evaluate it symbolically, either by clicking on the → button on the 'Symbolic' toolbar or by typing `[Ctrl].`, the keyboard alternative.

(a) $111\,111\,111\,111 + 333\,333\,333\,333$

(b) $111\,111\,111\,111 - 333\,333\,333\,333$

(c) $111\,111\,111\,111 \times 333\,333\,333\,333$

(d) $\dfrac{111\,111\,111\,111}{333\,333\,333\,333}$

(e) 2^{50} (f) $25!$ (g) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$ (h) $\operatorname{mod}(32, 6)$

(i) $\operatorname{mod}(2^{10\,000}, 10\,001)$

Comment

The answers provided by Mathcad are as follows.

(a) 444 444 444 444 (b) −222 222 222 222

(c) 37 037 037 036 962 962 962 963 (d) $\frac{1}{3}$ (e) 1 125 899 906 842 624

(f) 15 511 210 043 330 985 984 000 000 (g) $\frac{29}{20}$

(h) 2 (i) 4674

Factorisation

A positive integer can be expressed as a product of powers of prime numbers, by using the symbolic keyword 'factor'.

Activity 5.10 Factorisation

Enter each of the following integers, and apply the symbolic keyword 'factor' to it. To do this, either click on the 'factor' button on the 'Symbolic' toolbar, or type `[Ctrl][Shift].factor` .

In this activity, use the worksheet that you created for Activity 5.9.

(a) 10 001 (b) 3 220 422 643 (c) 50!

Comment

(a) $10\,001 = 73 \times 137$ (b) $3\,220\,422\,643 = 49\,139 \times 65\,537$

(c) $50! = 2^{47} \times 3^{22} \times 5^{12} \times 7^{8} \times 11^{4} \times 13^{3} \times 17^{2} \times 19^{2} \times 23^{2} \times 29 \times 31$
$\times 37 \times 41 \times 43 \times 47$

There is much interest in discovering large prime numbers. By applying symbolic evaluation in Mathcad to suitable large integers, large primes may be found. A useful rule of thumb here is that

> if an integer n has only small prime factors, then some nearby integer may well have large ones.

So it is a good idea to try factorising numbers of the forms $2^n \pm 1$, $3^n \pm 1$, etc. For example, applying the symbolic keyword 'factor' to $2^{45} + 1$ gives

$$2^{45} + 1 = 3^3 \times 11 \times 19 \times 331 \times 18\,837\,001;$$

hence 18 837 001 is prime.

If you are tempted to try calculations like these, then remember to save your work at each stage, because Mathcad may be unable to cope with huge calculations. (Mathcad's symbolic evaluations can handle numbers with about 65 000 digits.)

The largest known prime number (in October 2010) is $2^{43\,112\,609} - 1$, which has almost thirteen million digits. It is one of the family of Mersenne prime numbers. These are all of the form $2^p - 1$, where p is a prime number, and they are found by applying a special test to numbers of this form.

In general, it is much harder to factorise large numbers than it is to perform basic arithmetic operations with them. Some large numbers, like those of the form 10^n, factorise very quickly, but most do not. As you saw in Section 4 of the main text, this fact makes it possible to base ciphers on large numbers which are the product of two large prime numbers.

Save the worksheet that you have created, if you wish. Then close the file.

Chapter D3, Section 1
Symmetry

1.3 Using symmetries

This subsection will not be assessed.

In Section 1 of the main text, you saw how the pattern, or structure, of a plane set X can be described by finding its set of symmetries, $S(X)$. In the optional Mathcad files for this chapter you will see how knowing the symmetries can be helpful if you wish to plot the plane set using a computer package. These files are self-contained and may be worked through after reading the following introductory remarks.

First, in file 221D3-01, we explain how Mathcad can be used to plot a path made up of line segments placed end to end. Such a path is called a *piecewise linear path*, or PLP for short, and it is defined by prescribing a finite sequence of points in the plane, say $p(i)$ for $i = 0, 1, \ldots, n$, and then joining $p(0)$ to $p(1)$, $p(1)$ to $p(2)$, and so on, using line segments.

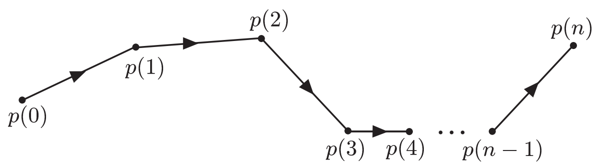
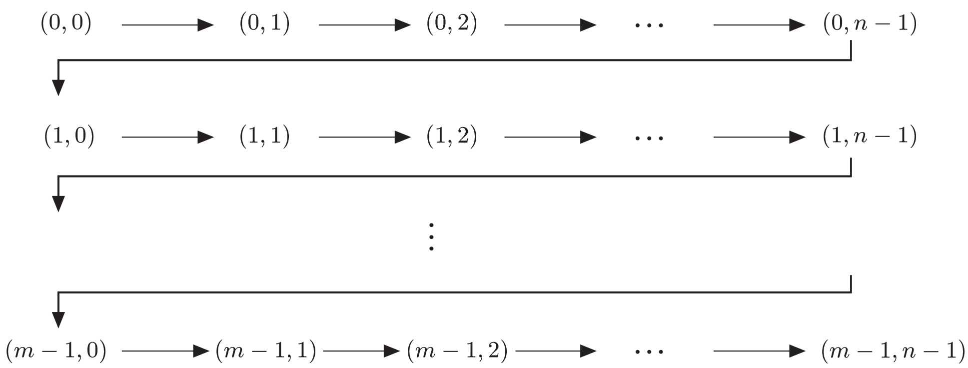


Figure 1.1 A piecewise linear path

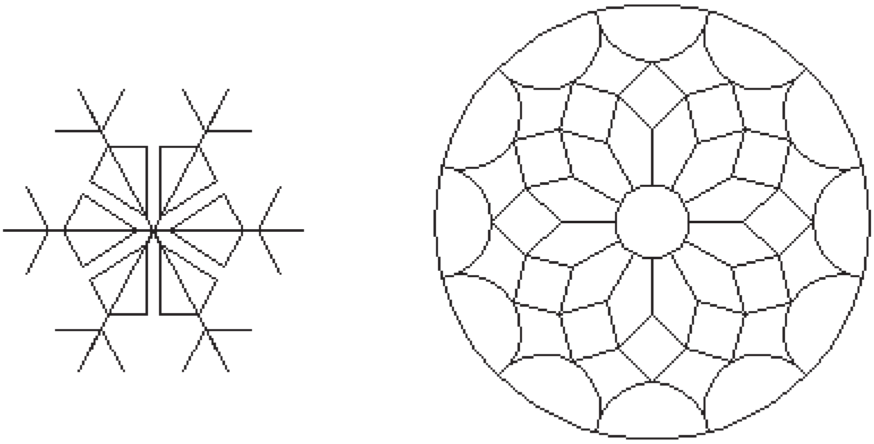
In this diagram, the arrows show the direction in which the line segments are plotted. Notice that a PLP with n line segments requires $n+1$ points, or *vertices*, to determine it.

One way to prescribe the vertices of a PLP in Mathcad is to take i to be the range variable $i := 0, 1 \,..\, n$, and define $p(i)$ to be a function of i whose values are vectors, that is, points in the plane. The corresponding PLP can then be plotted using an X-Y Plot, with the components $p(i)_0$ and $p(i)_1$ as the arguments on the x-axis and y-axis, respectively, and with the trace type set to 'lines'.

In file 221D3-02 Mathcad is used to plot a symmetric PLP corresponding to a vector-valued function $SP(j, i)$ of two range variables $j := 0, 1 \,..\, m-1$ and $i := 0, 1 \,..\, n-1$, by going through the range variables in the following order.



This facility makes it possible to plot plane sets with many symmetries, such as the snowflake and rose window shown below. This is done in files 221D3-02 and 221D3-03.



(a) (b)

Figure 1.2 (a) A snowflake (b) A rose window

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